Introduction

The geometry of circles, triangles and quadrilaterals represents some of the oldest mathematics know to man. At its origins, mathematics was a discovery, not an invention, in that mathematical questions rouse from inquiries into astronomy, geography, engineering, even astrology and alchemy. The prestigious scientists of the times would be called today interdisciplinary investigators, except all these fields had not yet consolidated as separate disciplined, and were fused into one complex approach to studying the world. While Euclid, for example (around 300 BC) was primarily a mathematician, and essentially the father of a few fields like conic geometry, and number theory, Pythagoras (around 500 BC) was primarily a philosopher, interested in ethics, politics and mysticism besides mathematics and Ptolemy (around 100 AD) was also a prominent astronomer, geographer and astrologer.

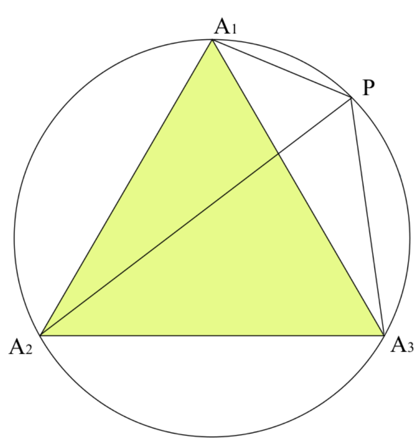
A lot of the beautiful and widely used results in geometry were forged in ancient times, often of obscure origin (while one may be able to locate references where a theorem was state and proved, it is harder to discard the possibility of earlier accounts, which did not get preserved by history. One must remember that these early works preceded most of the more modern mathematical formalizations, which came a lot later. For example, trigonometry did not yet exist even in the times of Ptolemy, hence results could not be formulated in terms of trigonometric functions, but instead were using arcs and "chords." A lot of alternative proofs to ancient results sprouted along history, making use of new developments and mathematical frameworks. Part of the attraction of coming up with new proofs of an already known result may be to illustrate a progressive timeline of mathematics, but also to unify the perspective over different fields.

Some history

In this paper, we generalize and discuss a result originally proved by Van Schooten, which in turn relates to Ptolemy's Theorem [1]. We present a few proofs: one based directly on the

A proof using Ptolemy's Theorem

Theorem $\mathbf{S_3}$. Consider $A_1A_2A_3$ an equilateral triangle, and consider P a point on the small arc $\overset{\frown}{PA_1}$ of its circumscribed circle. Then $PA_1 + PA_3 = PA_2$.



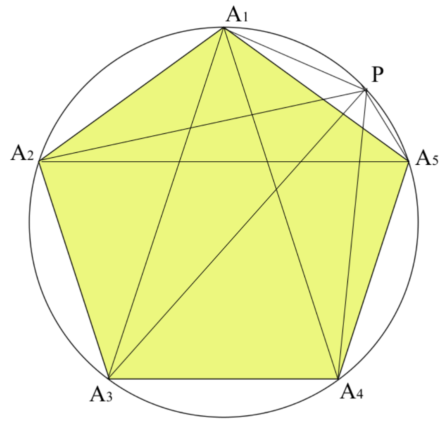
Proof. Since the quadrilateral $PA_1A_2A_3$ is inscribed in a circle, we can use Ptolemy's theorem:

$$PA_1 \cdot A_2A_3 + A_1A_2 \cdot PA_3 = PA_2 \cdot A_1A_3$$

But $A_1A2 = A_2A_3 = A_1A_3 = l$, hence $l \cdot (PA_1 + PA_3) = l \cdot PA_2 \implies PA_1 + PA_3 = PA_2$

□

Theorem $\mathbf{S_5}$. Consider $A_1A_2A_3A_4A_5$ a regular pentagon, and consider P a point on the small arc $\overset{\frown}{PA_1}$ of its circumscribed circle. Then $PA_1 + PA_3 + PA_5 = PA_2 + PA_4$.



Proof. To simplify notation, notice that $A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_1 = l$, that $A_1A_3 = A_2A_4 = A_3A_5 = A_4A_6 = A_5A_2 = d$. The quadrilaterals $PA_1A_2A_5$, $PA_1A_4A_5$, $PA_1A_2A_3$ and $PA_1A_3A_4$ are all inscribed in the circle, hence we can apply Ptolemy's theorem for each, and obtain:

- $PA_5 \cdot A_1A_2 + PA_1 \cdot A_2A_5 = PA_2 \cdot A_1A_5 \implies l \cdot PA_5 + d \cdot PA_1 = l \cdot PA_2$
- $PA_5 \cdot A_1A_4 + PA_1 \cdot A_4A_5 = PA_4 \cdot A_1A_5 \implies d \cdot PA_5 + l \cdot PA_1 = l \cdot PA_4$
- $PA_1 \cdot A_2A_3 + A_1A_2 \cdot PA_3 = A_1A_3 \cdot PA_2 \implies l \cdot PA_1 + l \cdot PA_3 = d \cdot PA_2$
- $PA_1 \cdot A_3A_4 + A_1A_3 \cdot PA_4 = A_1A_4 \cdot PA_3 \implies d \cdot PA_3 = l \cdot PA_1 + d \cdot PA_4$

Adding the equations side by side, it follows that $(d + l)(PA_1 + PA_3 + PA_5) = (d + l)(PA_2 + PA_4)$, hence:

$$PA_1 + PA_3 + PA_5 = PA_2 + PA_4$$

$\square$

A natural subsequent question is whether the theorem, found true for the $\mathbf{S_3}$ and $\mathbf{S_5}$ cases, applies in general for any $\mathbf{S_{2n+1}}$.

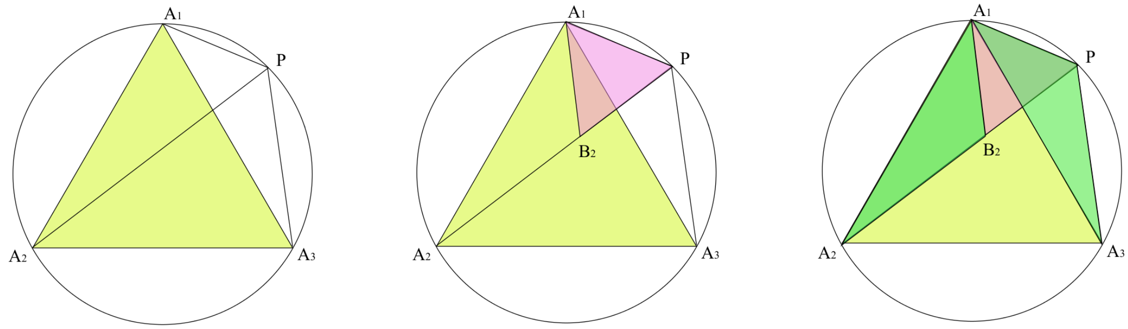
Theorem $\mathbf{S_{2n+1}}$. Consider $A_1 \dots A_{2n+1}$ a regular polygon, and consider P a point on the small arc $\overset{\frown}{PA_1}$ of its circumscribed circle. Then:

$$\sum_{j=0}^{n} PA_{2j+1} = \sum_{j=1}^{n} PA_{2j}$$

While a proof based directly on Ptolemy's theorem is somewhat tedious in the general case, below we present two different approaches, a geometric proof using an auxiliary construction, and an analytic proof using complex numbers.

A direct proof with congruence

Proof for $\mathbf{S_3}$. Recall that $A_1A_2A_3$ is an equilateral triangle, and that P is on its circumscribed circle, on the small arc $\overset{\frown}{A_1A_3}$. We want to prove that $PA_1 + PA_3 = PA_2$. Since $PA_1 < PA_2$, we can consider the point B_2 on the segment PA_2 such that $PB_2 = PA_1$. This creates the triangle PA_1B_2, which is isoceles (since $PA_1 = PB_2$), and has $\measuredangle A_1PB_2 = \dfrac{\overset{\frown}{A_1A_2}}{2} = \measuredangle A_1A_3A_2 = 60°$, hence is equilateral. This implies $PA_1 = PB_2 = A_1B_2$, and $\measuredangle PB_2A_1 = PA_1B_2 = A_1PB_2 = 60°$.

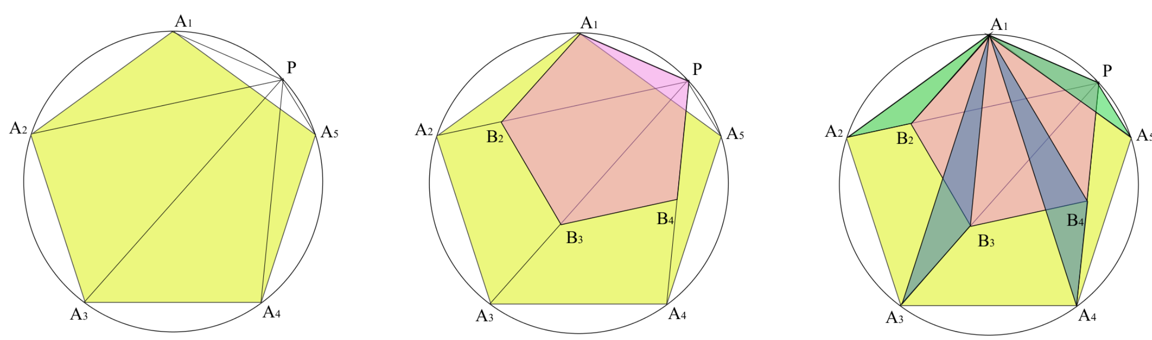


This construction creates a pair of congruent triangles: $\Delta A_1B_2A_2 \equiv \Delta A_1PA_2$. Indeed, $\measuredangle A_1A_2B_2 = \dfrac{\overset{\frown}{PA_1}}{2} = \measuredangle A_1A_3P$, and $\measuredangle A_1B_2A_2 = \measuredangle A_1PA_3 = 120°$. It follows that the remaining pair of angles are also equal. Also: $A_1A_2 = A_1A_3$. Hence $\Delta A_1B_2A_2 \equiv \Delta A_1PA_2$ (angle-side-angle criterion). It follows that $B_2A_2 = PA_3$. We have thus shown that $PA_2 = PB_2 + B_2A_2 = PA_1 + PA_3$.

The construction can be generalized for regular polygons with higher numbers of sides, as long as this number is odd. Below, we work out the cases of a pentagon and a heptagon, illustrating a strategy that clearly applies in general.

Proof for $\mathbf{S_5}$. $A_1 \ldots A_5$ is a regular pentagon and P is a point on the small arc $\overset{\frown}{A_1A_5}$. We want to prove that $PA_1 + PA_3 + PA_5 = PA_2 + PA_4$.

Consider points B_2, B_3 and B_4 on the segments PA_2, PA_3 and PA_4 respectively, so that the pentagon $PA_1B_2B_3B_4$ is regular. This is possible because $\measuredangle A_1PA_2 = \measuredangle A_2PA_3 = \measuredangle A_3PA_4 = 36°$.



This construction creates two pairs of congruent triangles: $\Delta A_1B_2A_2 \equiv \Delta A_1PA_5$ and $\Delta A_1B_3A_3 \equiv \Delta A_1B_4A_4$. Indeed: $\measuredangle A_1A_2B_2 = \dfrac{\overset{\frown}{A_1P}}{2} = \measuredangle A_1A_5P$, and $\measuredangle A_1B_2A_2 = \measuredangle A_1PA_5 = 144°$, implying that the third pair are also equal: $\measuredangle A_2A_1B_2 = \measuredangle A_5A_1P$. Moreover, $A_1A_2 = A_1A_5$ (as sides of the original regular pentagon). Hence $\Delta A_1B_2A_2 \equiv \Delta A_1PA_5$ (angle-side-angle). It follows directly that $B_2A_2 = PA_5$. Similarly: $\measuredangle A_1A_3B_3 = \frac{\overset{\frown}{A_1P}}{2} = \measuredangle A_1A_4B_4$ and $\measuredangle A_1B_3A_3 = \measuredangle A_1B_4A_4 = 144°$. This, together with $A_1A_3 = A_1A_4$ (as diagonals in the original pentagon), imply the congruence $\Delta A_1B_3A_3 \equiv \Delta A_1B_4A_4$ (angle-side-angle). It follows that $B_3A_3 = B_4A_4$. We now have the following:

- $PA_1 = PB_4$ (from the construction of our first regular pentagon of side PA_1)
- $PB_3 = PB_2$ (as equal diagonals of the pentagon above)
- $PA_5 = B_2A_2$ (from the congruence of the first triangle pair)

- $B_3A_3 = B_4A_4$ (from the congruence of the second triangle pair)

Hence: $PA_1 + (PB_3 + B_3A_3) + PA_5 = (PB_4 + B_4A_4) + (PB_2 + B_2A_2) \implies$

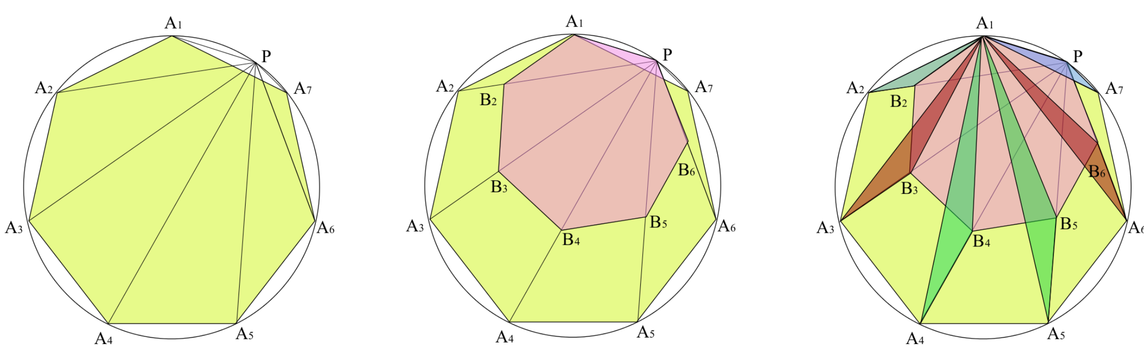
$$PA_1 + PA_3 + PA_5 = PA_2 + PA_4$$

□

Proof for S₇. $A_1 \dots A_7$ is a regular pentagon and P is a point on the small arc $\overset{\frown}{A_1A_7}$. We want to prove that $PA_1 + PA_3 + PA_5 + PA_7 = PA_2 + PA_4 + PA_6$.

Consider points B_j, for $j = \overline{2,6}$ on the corresponding PA_j segments, so that the heptagon $PA_1B_2 \dots B_6$ is regular. With this construction, we obtain three pairs of congruent triangles: $\Delta A_1B_2A_2 \equiv \Delta A_1PA_7$, $\Delta A_1B_3A_3 \equiv \Delta A_1B_6A_6$ and $\Delta A_1B_4A_4 \equiv A_1B_5A_5$. It follows that:

- $PA_1 = PB_6$ (by construction, as equal sides of a regular heptagon)
- $PB_5 = PB_2$ (equal diagonals in a regular heptagon)
- $PB_3 = PB_4$ (equal diagonals in a regular heptagon)
- $PA_7 = B_2A_2$ (from the congruence of the first triangle pair)
- $B_3A_3 = B_6A_6$ (from the congruence of the first triangle pair)
- $B_5A_5 = B_4A_4$ (from the congruence of the first triangle pair)



It follows that: $PA_1 + (PB_3 + B_3A_3) + (PB_5 + B_5A_5) + PA_7 = (PB_2 + B_2A_2) + (PB_4 + B_4A_4) + PA_6 \implies$

$$PA_1 + PA_3 + PA_5 + PA_7 = PA_2 + PA_4 + PA_6$$

□

A proof using complex numbers

Call $\xi_1, \xi_2, \dots, \xi_N$ the roots fo unity of odd order $N = 2n + 1$:

$$\xi_k = e^{i\theta_k} = \cos(\theta_k) + i\sin(\theta_k) = \xi_1^k, \text{ for all } k = \overline{1, 2n+1}$$

where $\theta_k = \dfrac{2\pi k}{2n+1} = k\theta_1$.

Lemma. For any arbitrary $\theta \in [0, \theta_1]$, we have that:

$$\sum_{k=1}^{2n+1} (-1)^k \left|e^{i\theta} - e^{i\theta_k}\right| = 0 \tag{1}$$

Remark. This result does not hold for even roots of unity.

Proof. It is easy to show that:

$$\begin{aligned}
\left|e^{i\theta} - e^{i\theta_k}\right|^2 &= (\cos\theta - \cos\theta_k)^2 + (\sin\theta - \sin\theta_k)^2 = 2 - 2(\cos\theta\cos\theta_k + \sin\theta\sin\theta_k) \\
&= 2 - 2\cos(\theta_k - \theta) = 4\sin\left(\frac{\theta_k - \theta}{2}\right)
\end{aligned}$$

Hence: $|e^{i\theta} - e^{i\theta_k}| = 2\left|\sin\frac{\theta_k - \theta}{2}\right|$. Since $0 \le \theta \le \theta_k$, it follows that $0 \le \theta_k - \theta \le 2\pi$, so $0 \le \frac{\theta_k - \theta}{2} \le \pi$, and $\sin\left(\frac{\theta_k - \theta}{2}\right) \ge 0$. This means that:

$$\left|e^{i\theta} - e^{i\theta_k}\right| = 2\sin\frac{\theta_k - \theta}{2} = 2\left(\sin\frac{\theta_k}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta_k}{2}\right)$$

Then the sum in (1) then becomes:

$$\sum_{k=1}^{2n+1} (-1)^k \left|e^{i\theta} - e^{i\theta_k}\right| = 2\cos\frac{\theta}{2}\sum_{k=1}^{2n+1} (-1)^k \sin\frac{\theta_k}{2} - 2\sin\frac{\theta}{2}\sum_{k=1}^{2n+1} (-1)^k \cos\frac{\theta_k}{2}$$

Consider the root of unity of order $2N$: $\psi = \cos\frac{\theta_1}{2} + i\sin\frac{\theta_1}{2}$. Hence: $\psi^k = \cos\frac{\theta_k}{2} + i\sin\frac{\theta_k}{2}$. Then:

$$\sum_{k=1}^{2n+1} (-1)^k \psi^k = -\psi \cdot \sum_{k=0}^{2n} (-\psi)^k = -\psi \cdot \frac{1 + \psi^{2n+1}}{1 - \psi}$$

But $\psi^{2n+1} = \cos\pi + i\sin\pi = -1$, hence $\sum_{k=1}^{2n+1} (-1)^k \psi^k = 0$, which means that its real and imaginary parts are both zero. In other words, both

$$\sum_{k=1}^{2n+1} (-1)^k \cos\frac{\theta_k}{2} = 0 \qquad \text{and} \qquad \sum_{k=1}^{2n+1} (-1)^k \sin\frac{\theta_k}{2} = 0$$

hence $\sum_{k=1}^{2n+1} (-1)^k \left|e^{i\theta} - e^{i\theta_k}\right| = 0$.

□

Returning now to our geometry problem, consider any regular polygon with $N = 2n+1$ vertices $A_1 \ldots A_{2n+1}$. WLOG, the polygon can be transformed (through shifts, rotations and scaling) so that the vertices correspond to the $2n+1$ roots of unity. Then our identity follows directly from Lemma 1:

$$\sum_{j=0}^{n} PA_{2j+1} = \sum_{j=1}^{n} PA_{2j}$$

References

[1] MAB Deakin. History of mathematics: Ptolemy's theorem. *Parabola*, 43(1):1–7, 2007.