# Chapter 10

Two-Sample Tests

## Learning Objectives

In this chapter, you learn how to use hypothesis testing for comparing the difference between:

- The means of two independent populations
- The means of two related populations
- The proportions of two independent populations
- The variances of two independent populations by testing the ratio of the two variances

## Two-Sample Tests

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1 vs. Group 2</td>
<td>Same group before vs. after treatment</td>
<td>Proportion 1 vs. Proportion 2</td>
<td>Variance 1 vs. Variance 2</td>
</tr>
</tbody>
</table>

## Difference Between Two Means

**Population means, independent samples**

- Goal: Test hypothesis or form a confidence interval for the difference between two population means, $\mu_1 - \mu_2$

- The point estimate for the difference is $\bar{X}_1 - \bar{X}_2$

- $\sigma_1$ and $\sigma_2$ unknown, assumed equal
- $\sigma_1$ and $\sigma_2$ unknown, not assumed equal

## Hypothesis Tests for Two Population Means

- **Lower-tail test:**
  - $H_0: \mu_1 \geq \mu_2$
  - $H_1: \mu_1 < \mu_2$
  - i.e., $\mu_1 - \mu_2 \leq 0$

- **Upper-tail test:**
  - $H_0: \mu_1 \leq \mu_2$
  - $H_1: \mu_1 > \mu_2$
  - i.e., $\mu_1 - \mu_2 > 0$

- **Two-tail test:**
  - $H_0: \mu_1 = \mu_2$
  - $H_1: \mu_1 \neq \mu_2$
  - i.e., $\mu_1 - \mu_2 = 0$
Hypothesis tests for $\mu_1 - \mu_2$

Two Population Means, Independent Samples

- **Lower-tail test:**
  - $H_0: \mu_1 - \mu_2 \leq 0$
  - $H_1: \mu_1 - \mu_2 > 0$

- **Upper-tail test:**
  - $H_0: \mu_1 - \mu_2 \geq 0$
  - $H_1: \mu_1 - \mu_2 < 0$

- **Two-tail test:**
  - $H_0: \mu_1 - \mu_2 = 0$
  - $H_1: \mu_1 - \mu_2 \neq 0$

- **The test statistic is:**
  $$t_{\text{STAT}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- **Where $t_{\text{STAT}}$ has d.f. = $n_1 + n_2 - 2$**

- **Assumptions:**
  - Samples are randomly and independently drawn
  - Populations are normally distributed or both sample sizes are at least 30
  - Population variances are unknown but assumed equal

### Pooled-Variance t Test Example

You are a financial analyst for a brokerage firm. Is there a difference in mean yield ($\alpha = 0.05$)?

Assuming both populations are approximately normal with equal variances, is there a difference in mean yield ($\alpha = 0.05$)?

<table>
<thead>
<tr>
<th>Sample</th>
<th>Mean</th>
<th>Std Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>NYSE</td>
<td>1.30</td>
<td>1.16</td>
</tr>
<tr>
<td>NASDAQ</td>
<td>3.27</td>
<td>2.53</td>
</tr>
</tbody>
</table>

**Calculating the Test Statistic**

$$t_{\text{STAT}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}}} = \frac{3.27 - 2.53}{\sqrt{\frac{1.5021}{21} + \frac{1.5021}{25}}} = 2.040$$
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Hypothesis Test Solution

\[ H_0: \mu_1 - \mu_2 = 0 \text{ i.e. } (\mu_1 = \mu_2) \]

\[ \text{df} = 21 + 25 - 2 = 44 \]

Critical Values: \( t = 2.0154 \)

Test Statistic:

\[ t_{\text{stat}} = \frac{3.27 - 2.53}{\sqrt{\left(\frac{1}{21} \cdot 1.16 \cdot 0.23\right)}} = 2.040 \]

Decision:

Reject \( H_0 \) at \( \alpha = 0.05 \)

Conclusion: There is evidence of a difference in means.

Confidence Interval for \( \mu_1 - \mu_2 \)

Since we rejected \( H_0 \) can we be 95% confident that \( \mu_{\text{NYSE}} > \mu_{\text{NASDAQ}} \)?

95% Confidence Interval for \( \mu_{\text{NYSE}} - \mu_{\text{NASDAQ}} \)

\[
(3.27 - 2.53) \pm 2.015(1.16)(0.23) = (0.009, 1.471)
\]

Since 0 is less than the entire interval, we can be 95% confident that \( \mu_{\text{NYSE}} > \mu_{\text{NASDAQ}} \)

Excel Pooled-Variance t Test Comparing NYSE & NASDAQ

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Minitab Pooled-Variance t Test Comparing NYSE & NASDAQ

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(0.009, 1.471)
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Since 0 is less than the entire interval, we can be 95% confident that \( \mu_{\text{NYSE}} > \mu_{\text{NASDAQ}} \)

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed or both sample sizes are at least 30
- Population variances are unknown and cannot be assumed to be equal

The test statistic is:

\[
t_{\text{stat}} = \frac{(X_1 - X_2) - (\mu_1 - \mu_2)}{S_1^2 \cdot \frac{1}{n_1} + S_2^2 \cdot \frac{1}{n_2}}
\]

\( \sigma_1^2 \) and \( \sigma_2^2 \) unknown and not assumed equal
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10-4

Prof. Shuguang Liu

Related Populations

The Paired Difference Test

Tests Means of 2 Related Populations

Related samples

Paired or matched samples
Repeated measures (before/after)
Use difference between paired values:

- Eliminates Variation Among Subjects
- Both Populations Are Normally Distributed
- Or, if not Normal, use large samples

The Paired Difference Test:
Finding \( t_{\text{STAT}} \)

- The test statistic for \( \mu_D \) is:

\[
   t_{\text{STAT}} = \frac{\bar{D} - \mu_D}{S_D / \sqrt{n}}
\]

Where \( t_{\text{STAT}} \) has \( n - 1 \) d.f.

The Paired Difference Test: Possible Hypotheses

Paired Samples

Lower-tail test:
- \( H_0: \mu_D \geq 0 \)
- \( H_1: \mu_D < 0 \)

Upper-tail test:
- \( H_0: \mu_D \leq 0 \)
- \( H_1: \mu_D > 0 \)

Two-tail test:
- \( H_0: \mu_D = 0 \)
- \( H_1: \mu_D \neq 0 \)

Reject \( H_0 \) if \( t_{\text{STAT}} < t_{\alpha/2} \)
Reject \( H_0 \) if \( t_{\text{STAT}} > t_{\alpha/2} \)

Where \( t_{\text{STAT}} \) has \( n - 1 \) d.f.

Related Populations

The Paired Difference Test

The paired difference is \( D_i \), where

\[
   D_i = X_{1i} - X_{2i}
\]

The point estimate for the paired difference population mean \( \mu_D \) is \( \bar{D} \):

\[
   \bar{D} = \frac{\sum D_i}{n}
\]

The sample standard deviation is \( S_D \):

\[
   S_D = \sqrt{\frac{\sum (D_i - \bar{D})^2}{n-1}}
\]

\( n \) is the number of pairs in the paired sample

Related Populations

The Paired Difference Test

The Paired Difference Confidence Interval

The confidence interval for \( \mu_D \) is

\[
   \bar{D} \pm t_{\alpha/2} \frac{S_D}{\sqrt{n}}
\]

where

\[
   S_D = \sqrt{\frac{\sum (D_i - \bar{D})^2}{n-1}}
\]

Assume you send your salespeople to a "customer service" training workshop. Has the training made a difference in the number of complaints? You collect the following data:

<table>
<thead>
<tr>
<th>Salesperson</th>
<th>Number of Complaints:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Before (1)</td>
</tr>
<tr>
<td>C.B.</td>
<td>6</td>
</tr>
<tr>
<td>T.F.</td>
<td>20</td>
</tr>
<tr>
<td>M.H.</td>
<td>3</td>
</tr>
<tr>
<td>R.K.</td>
<td>0</td>
</tr>
<tr>
<td>M.O.</td>
<td>4</td>
</tr>
</tbody>
</table>

\[
   \bar{D} = \frac{\sum D_i}{n} = -4.2
\]

\[
   S_D = \sqrt{\frac{\sum (D_i - \bar{D})^2}{n-1}} = 5.67
\]

Paired Difference Test: Example

\[
   D = \frac{\sum D_i}{n}
\]
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Paired Difference Test:

**Solution**

- Has the training made a difference in the number of complaints (at the 0.01 level)?

\[
\begin{align*}
H_0 &: \mu = 0 \\
H_1 &: \mu \neq 0
\end{align*}
\]

\[\alpha = 0.01\]

\[d = 4.2\]

\[t_{0.005} = \pm 4.604\]

\[d.f. = n - 1 = 4\]

**Decision:** Do not reject \(H_0\) (\(t_{\text{stat}}\) is not in the reject region)

**Conclusion:** There is insufficient evidence there is significant change in the number of complaints.

**Paired t Test In Excel**

**Paired T-Test and CI:** After, Before

<table>
<thead>
<tr>
<th></th>
<th>After</th>
<th>Before</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Mean</td>
<td>2.40</td>
<td>6.60</td>
</tr>
<tr>
<td>StdDev</td>
<td>2.61</td>
<td>7.80</td>
</tr>
<tr>
<td>SE Mean</td>
<td>1.17</td>
<td>3.49</td>
</tr>
</tbody>
</table>

95% CI for mean difference: (-11.25, 2.85)

T-Test of mean difference = 0 (vs not = 0): T-Value = -1.66 P-Value = 0.173

**Two Population Proportions**

**Goal:** test a hypothesis or form a confidence interval for the difference between two population proportions, \(\pi_1 - \pi_2\)

The point estimate for the difference is \(\hat{p}_1 - \hat{p}_2\)

In the null hypothesis we assume the null hypothesis is true, so we assume \(\pi_1 = \pi_2\) and pool the two sample estimates

The pooled estimate for the overall proportion is:

\[
\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}
\]

where \(X_1\) and \(X_2\) are the number of items of interest in samples 1 and 2

\[Z_{\text{STAT}} = \left(\frac{p_1 - p_2}{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right)
\]

where \(p = \frac{X_1 + X_2}{n_1 + n_2}\), \(p_1 = \frac{X_1}{n_1}\), \(p_2 = \frac{X_2}{n_2}\)

Two Population Proportions

(continued)
Hypothesis Test Example: 
Two population Proportions

Is there a significant difference between the proportion of critical values indicated they would vote Yes between men and the proportion of women who will vote Yes on Proposition A? 

- In a random sample, 36 of 72 men and 35 of 50 women indicated they would vote Yes 
- Test at the .05 level of significance

The pooled estimate for the overall proportion is:

\[ \hat{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{36 + 35}{72 + 50} = \frac{71}{122} = 0.582 \]

Hypothesis Test Example: 
Two population Proportions

The hypothesis test is:

- The hypothesis test is:
  - Upper-tail test:
    \[ H_0: \pi_1 - \pi_2 = 0 \] (the two proportions are equal)
    \[ H_1: \pi_1 - \pi_2 > 0 \] (there is a significant difference between proportions)
  - Two-tail test:
    \[ H_0: \pi_1 = \pi_2 \] (the two proportions are equal)
    \[ H_1: \pi_1 \neq \pi_2 \] (there is a significant difference between proportions)

The sample proportions are:

- Men: \( p_1 = 36/72 = 0.50 \)
- Women: \( p_2 = 35/50 = 0.70 \)

The test statistic for \( x_1 - x_2 \) is:

\[ z = \frac{\hat{p} - \pi_0}{\sqrt{\frac{1}{n_1} \frac{1}{n_2}}} \]

Decision: Reject \( H_0 \) if \( z_s > 1.96 \) or \( z_s < -1.96 \)

Conclusion: There is evidence of a difference in proportions who will vote Yes between men and women.

Two Proportion Test In Excel

- Since \( -2.20 < -1.96 \)
- Or
- Since \( p-value = 0.028 < 0.05 \)
- We reject the null hypothesis

Conclusion: There is evidence of a difference in proportions who will vote yes between men and women.
Two Proportion Test in Minitab Shows The Same Conclusions

<table>
<thead>
<tr>
<th>Sample</th>
<th>X</th>
<th>N</th>
<th>Sample p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>36</td>
<td>72</td>
<td>0.500000</td>
</tr>
<tr>
<td>2</td>
<td>35</td>
<td>50</td>
<td>0.700000</td>
</tr>
</tbody>
</table>

Difference = p (1) - p (2)

Estimate for difference: -0.2

95% CI for difference: (-0.371676, -0.028324)

Test for difference = 0 (vs not = 0): Z = -2.28  P-Value = 0.022

Confidence Interval for Two Population Proportions

\[
\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}
\]

The F Distribution

- The F critical value is found from the F table
- There are two degrees of freedom required: numerator and denominator
- The larger sample variance is always the numerator
- When \( F_{STAT} = \frac{S_1^2}{S_2^2} \) \( df_1 = n_1 - 1 \); \( df_2 = n_2 - 1 \)
- In the F table, the numerator degrees of freedom determine the column, and denominator degrees of freedom determine the row

F Test: An Example

You are a financial analyst for a brokerage firm. You want to compare dividend yields between stocks listed on the NYSE & NASDAQ. You collect the following data:

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
<th>NASDAQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>21</td>
<td>25</td>
</tr>
<tr>
<td>Mean</td>
<td>3.27</td>
<td>2.53</td>
</tr>
<tr>
<td>Std dev</td>
<td>1.30</td>
<td>1.16</td>
</tr>
</tbody>
</table>

Is there a difference in the variances between the NYSE & NASDAQ at the \( \alpha = 0.05 \) level?
F Test: Example Solution

- Form the hypothesis test:
  \[ H_0: \sigma_1^2 = \sigma_2^2 \] (there is no difference between variances)
  \[ H_a: \sigma_1^2 \neq \sigma_2^2 \] (there is a difference between variances)

- Find the F critical value for \( \alpha = 0.05 \):
  \[ \text{Numerator d.f.} = n_1 - 1 = 21 - 1 = 20 \]
  \[ \text{Denominator d.f.} = n_2 - 1 = 25 - 1 = 24 \]
  \[ F_{\alpha/2} = F_{0.025, 20, 24} = 2.33 \]

- The test statistic is:
  \[ F_{\text{STAT}} = \frac{S_1^2}{S_2^2} = \frac{1.3^2}{1.1^2} = 1.256 \]

- Conclusion:
  There is insufficient evidence of a difference in variances at \( \alpha = .05 \) because:
  \[ F_{\text{STAT}} = 1.256 < 2.327 = F_{0.05/2, 20, 24} \]
  or
  \[ p\text{-value} = 0.589 > 0.05 = \alpha. \]

Two Variance F Test In Excel

Conclusion: There is insufficient evidence of a difference in variances at \( \alpha = .05 \) because:

\[ F_{\text{STAT}} = 1.256 < 2.327 = F_{0.05/2, 20, 24} \] or
\[ p\text{-value} = 0.589 > 0.05 = \alpha. \]

Chapter Summary

- Compared two independent samples
  - Performed pooled-variance t test for the difference in two means
  - Performed separate-variance t test for difference in two means
  - Formed confidence intervals for the difference between two means
- Compared two related samples (paired samples)
  - Performed paired t test for the mean difference
  - Formed confidence intervals for the mean difference
- Compared two population proportions
  - Formed confidence intervals for the difference between two population proportions
  - Performed Z-test for two population proportions
  - Performed F test for the ratio of two population variances