Some remarks on a paper by L. Carlitz

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Abstract

We study a family of orthogonal polynomials which generalizes a sequence of polynomials considered by L. Carlitz. We show that they are a special case of the Sheffer polynomials and point out some interesting connections with certain Sobolev orthogonal polynomials.

Keywords: Sheffer polynomials, Favard theorem, Sobolev orthogonal polynomials, generating functions, hypergeometric polynomials, special functions.

MSC-class: 33A65 (Primary) 11B83, 46E39 (Secondary)

1 Introduction

In [32] R. Kelisky defined a set of rational integers $T_n$ by means of the exponential generating function

$$
\exp \left[ \arctan(z) \right] = \sum_{n=0}^{\infty} T_n \frac{z^n}{n!}.
$$

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L. Carlitz extended Kelisky’s idea by defining the polynomials $T_n(x)$, with

$$\exp \left[ x \arctan (z) \right] = \sum_{n=0}^{\infty} T_n(x) \frac{z^n}{n!}.$$  \hspace{1cm} (1)

In his article [15], he showed that $T_n(x)$ satisfies the recurrence

$$T_{n+1} = xT_n - n(n-1)T_{n-1}, \quad n \geq 0,$$  \hspace{1cm} (2)

where $T_{-1}(x) = 0$ and $T_{0}(x) = 1$. He also proved [16] several arithmetic properties of $T_n(x)$.

In [31] S. Kajijer considered the recurrence (2) and concluded that:

1. The function $T_n(x)$ is a monic polynomial of degree $n$.

2. The set $\left\{ \frac{1}{n!} T_n(x) \mid n \geq 0 \right\}$ is an orthonormal basis of the Hilbert space $H^2(S, P)$, where $S = \{ z \in \mathbb{C} \mid \text{Im}(z) \in [-1, 1] \}$ and $P$ is the Poisson measure for 0.

3. The norm of the polynomial $\frac{1}{n!} T_n(x)$ is $\sqrt{2}$ if $n \geq 1$ and 1 if $n = 0$.

4. The exponential generating function of $T_n(x)$ is $\exp \left[ x \arctan (z) \right]$.

The same results and some extensions were presented by T. Araaya in [6]. He also showed that

$$\frac{1}{n!} T_n(x) = \frac{x}{n} P_{n-1}^{(1)} \left( \frac{x}{2}, \frac{\pi}{2} \right), \quad n \geq 1$$  \hspace{1cm} (3)

where $P_n^{(\lambda)} (x; \phi)$ is the Meixner-Pollaczek polynomial [17].

In this paper we extend (1) by consider the following problem.

**Problem 1** For which functions $f(z)$, will the polynomials $\Psi_n(x)$, generated by

$$\exp \left[ x f(z) \right] = \sum_{n=0}^{\infty} \Psi_n(x) \frac{z^n}{n!},$$  \hspace{1cm} (4)

form an orthogonal set?
In Section 2 we show that the, somewhat surprising, answer is that the function \( f(z) \) must be of the \( \arctan(x) \) type.

Problem 1 is a particular case of the Sheffer problem [57] of characterizing the orthogonal polynomials \( S_n(x) \) generated by

\[
F(z) \exp \left[ xf(z) \right] = \sum_{n=0}^{\infty} \Psi_n(x) \frac{z^n}{n!}
\]

where \( F(0) = 1 \) and \( f(0) = 0 \). Although the Sheffer polynomials have been studied extensively [3], [10], [19], [20], [26], [55] [56], [58], the limiting case (4) with \( F(z) = 1 \) does not appear to have been considered before.

It is known [2], [17], that the Sheffer polynomials reduce to the Hermite, Laguerre, Charlier, Meixner or Meixner-Pollaczek polynomials, depending on the choice of \( F(z) \) and \( f(z) \). Motivated by (3), we wondered if some of the polynomials generated by (4) will form a new class. In Section 3 we show that they will be limiting cases of the Laguerre, Meixner-Pollaczek or Meixner polynomials.

An interesting property of these polynomials is that they do not form an orthogonal set with respect to the standard inner-products, but they are orthogonal with respect to some new inner-products involving differential or difference operators. These classes of polynomials (called Sobolev orthogonal polynomials) have been the object of much attention in the last years (see [4], [11], [12], [14], [18], [25], [27], [28], [29], [30], [33], [37], [39], [41], [44], [45], [46], [47], [49], [52]).

2 The polynomials \( \Psi_n(x) \)

Definition 2 A function \( \mu(x) : R \to R \) is called a distribution function if:

1. \( \mu(x) \) is non-decreasing.
2. \( \mu(x) \) is bounded.
3. \( \mu(x) \) has finite moments, i.e., \( \int_{-\infty}^{\infty} x^n d\mu < \infty, \quad n \geq 0 \).

The set

\[
\mathcal{S}(\mu) = \{ x \in \mathbb{R} \mid \mu(x^+) - \mu(x^-) > 0 \}
\]

is called the spectrum of \( \mu(x) \).
We remind the reader of the following result, a proof of which can be found in [59].

**Theorem 3** A set of orthogonal polynomials \( \{p_n(x) \mid n \geq 0\} \) satisfies a three term recurrence relation of the form

\[
p_{n+1} = (A_n x + B_n) p_n - C_n p_{n-1}, \quad n \geq 0 \tag{5}
\]

with \( p_{-1}(x) = 0 \) and \( p_0(x) = 1 \).

The converse is known as Favard’s Theorem [24].

**Theorem 4** Let \( \{p_n(x) \mid n \geq 0\} \) be a sequence of polynomials, satisfying (5) with \( A_n \neq 0 \) for all \( n \geq 0 \). Then,

\( \text{(a)} \) There exists a function \( \mu(x) \) of bounded variation on \( (-\infty, \infty) \) such that

\[
\int_{-\infty}^{\infty} p_n(x)p_m(x)d\mu = M_n \delta_{n,m}, \quad n, m \geq 0
\]

where

\[
M_n = \mu_0 \frac{A_n}{A_0} \prod_{k=1}^{n} C_k
\]

and \( \mu_0 > 0 \) is a normalization constant.

\( \text{(b)} \) \( \mu(x) \) can be chosen to be real valued if and only if \( A_n, B_n \) and \( C_n \in \mathbb{R} \) for all \( n \geq 0 \).

\( \text{(c)} \) \( \mu(x) \) is a distribution with an infinite spectrum if and only if \( A_n, B_n \) and \( C_n \in \mathbb{R} \) for all \( n \geq 0 \) and \( M_n > 0 \) for all \( n \geq 1 \).

**Proof.** See [17] for a proof and [1, 13, 22, 42], for some generalizations.

Therefore, if \( \{\Psi_n(x) \mid n \geq 0\} \) is to be an orthogonal set, the polynomials \( \Psi_n(x) \) must satisfy a recurrence relation of the form

\[
\Psi_{n+1} = (A_n x + B_n) \Psi_n - C_n \Psi_{n-1} \tag{6}
\]

with \( \Psi_{-1}(x) = 0, \, \Psi_0(x) = 1 \).
Let’s define the function

\[ G(x, z) = \exp \left[ x f(z) \right]. \]  

From (4) we see that in order to have \( \Psi_0(x) = 1 \), we need \( f(0) = 0 \). Also, from

\[ \frac{\partial G}{\partial z} = x f'(z) G(x, z) = \sum_{n=0}^{\infty} \Psi_{n+1}(x) \frac{z^n}{n!}, \]

we have \( f'(0) x = \Psi_1(x) \). If \( \Psi_1(x) \) is to be a polynomial of degree 1, we need \( f'(0) \neq 0 \).

Taking the \( n^{th} \) derivative with respect to \( z \) in (8) and using Leibniz’s rule, we get

\[ \frac{\partial^{n+1} G}{\partial z^{n+1}}(x, z) = x \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(z) \frac{\partial^k G}{\partial z^k}(x, z), \quad n \geq 0. \]  

Setting \( z = 0 \) in (9) and using (4) we obtain

\[ \Psi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)}(0) \Psi_k(x) \]

\[ = x f'(0) \Psi_n(x) + x \sum_{k=0}^{n-1} \binom{n}{k} f^{(n+1-k)}(0) \Psi_k(x). \]

Comparing (10) with (6) we conclude that

\[ A_n = f'(0) \equiv a \]

and

\[ B_n \Psi_n - C_n \Psi_{n-1} = x \sum_{k=0}^{n-1} \binom{n}{k} f^{(n+1-k)}(0) \Psi_k(x). \]  

Using (10) in (12) we have

\[ B_n x \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(n-k)}(0) \Psi_k(x) - C_n x \sum_{k=0}^{n-2} \binom{n-2}{k} f^{(n-k-1)}(0) \Psi_k(x) \]

\[ = x \sum_{k=0}^{n-1} \binom{n}{k} f^{(n+1-k)}(0) \Psi_k(x), \]
which implies
\[ B_n f'(0) \Psi_{n-1}(x) = n f''(0) \Psi_{n-1}(x) \]
or
\[ B_n = n \frac{f''(0)}{f'(0)} \equiv nb. \] (14)

Rearranging terms in (13) and using (14), we get
\[ \sum_{k=0}^{n-2} \left[ nb \binom{n-1}{k} f^{(n-k)}(0) - C_n \binom{n-2}{k} f^{(n-k-1)}(0) - \binom{n}{k} f^{(n+1-k)}(0) \right] \Psi_k(x) = 0 \]
which gives
\[ \frac{-C_n}{n(n-1)} = \frac{f^{(k+1)}(0) - kbf^{(k)}(0)}{k(k-1)f^{(k-1)}(0)}. \] (15)

For equation (15) to be valid for \( 2 \leq k \leq n, \ n \geq 2 \), it is necessary that both sides be equal to the same constant \(-c\), with \( c > 0 \). Thus,
\[ \frac{-C_n}{n(n-1)} = -c \]
or
\[ C_n = cn(n-1), \quad c > 0 \] (16)

and
\[ f^{(k+1)}(0) - kbf^{(k)}(0) + ck(k-1)f^{(k-1)}(0) = 0, \quad k \geq 1. \] (17)

The solution of the recurrence (17), with \( f^{(1)}(0) = a \), is given by
\[ f^{(k)}(0) = \frac{ac}{R^-(bR^+ - 2c)} \Gamma(k) \left[ (R^+)^k - (R^-)^k \right], \quad k \geq 1 \] (18)
where \( \Gamma(\cdot) \) is the Gamma function and
\[ R^\pm = \frac{1}{2} \left[ b \pm \sqrt{b^2 - 4c} \right]. \] (19)

From (18) and \( f(0) = 0 \), we get
\[ f(z) = \frac{a}{\sqrt{b^2 - 4c}} \ln \left[ \frac{1 - zR^-}{1 - zR^+} \right] \] (20)
or
\[ f(z) = \frac{2a}{\sqrt{b^2 - 4c}} \left[ \text{arctanh} \left( \frac{b - 2cz}{\sqrt{b^2 - 4c}} \right) - \text{arctanh} \left( \frac{b}{\sqrt{b^2 - 4c}} \right) \right]. \]

We summarize the results of this section in the following theorem.
Theorem 5 If the family of polynomials \( \{ \Psi_n(x) \mid n \geq 0 \} \) defined by (4) satisfies (6), then

\[
f(z) = \frac{2a}{\sqrt{b^2 - 4c}} \left[ \text{arctanh} \left( \frac{b - 2cz}{\sqrt{b^2 - 4c}} \right) - \text{arctanh} \left( \frac{b}{\sqrt{b^2 - 4c}} \right) \right],
\]

and

\[
A_n = a, \quad B_n = nb, \quad C_n = cn(n - 1)
\]

with \( a \neq 0 \) and \( c > 0 \). The recurrence relation (6) takes the form

\[
\Psi_{n+1} = (ax + nb) \Psi_n - cn(n - 1) \Psi_{n-1}, \quad n \geq 0.
\]

Remark 6 In the remainder of the paper we shall stress the dependence of \( \Psi_n(x) \) on \( a, b \) and \( c \) by writing \( \Psi_n(x) = \Psi_n(x; a, b, c) \). Note that from (21) it follows that

\[
T_n(x) = \Psi_n(x; 1, 0, 1).
\]

Remark 7 Since \( C_1 = 0 \), it follows from Favard’s theorem that the full set \( \{ \Psi_n(x) \mid n \geq 0 \} \) is not orthogonal with respect to an inner-product generated by a distribution function. However, we shall see in the next section that \( \{ \Psi_n(x) \mid n \geq 1 \} \) is.

3 Representation of \( \Psi_n \)

In order to find a representation of \( \Psi_n(x; a, b, c) \), in terms of the classic hypergeometric polynomials, we shall consider three different cases, depending on the sign of the discriminant in (19).

3.1 Case 1: \( b^2 - 4c = 0, \; b \neq 0 \)

From (19), we have \( R^+ = R^- = \frac{1}{2} b \). Taking the limit as \( c \to \frac{b^4}{4} \) in (20) gives

\[
f(z) = -\frac{2a}{b} \frac{b_z}{\frac{b}{2}z - 1}.
\]

The generating function for the Laguerre polynomials is given by [9]

\[
(1 - z)^{-a-1} \exp \left[ x \frac{z}{z - 1} \right] = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n, \quad \alpha > -1.
\]

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Hence, from (24) and (25), we obtain
\[ \Psi_n(x; a, b, \frac{b^4}{4}) = n! \left( \frac{b}{2} \right)^n \lim_{a \to -1} L_{\alpha}^{(n)} \left( -\frac{2a}{b} x \right). \] (26)

To find the limit in (26) we prove the following lemma.

**Lemma 8** Let \( _rF_s \) be the hypergeometric function \([40]\). Then,
\[
\lim_{\omega \to 0} (\omega)_n \ _rF_s \left( \begin{array}{c} a_1(\omega), \ldots, a_r(\omega) \\ \omega, b_2(\omega), \ldots, b_s(\omega) \end{array} \bigg| x \right) = x \Gamma(n) \prod_{j=1}^{r} \frac{a_j(0)}{b_j(0)} \ _rF_s \left( \begin{array}{c} a_1(0) + 1, \ldots, a_r(0) + 1 \\ 2, b_2(0) + 1, \ldots, b_s(0) + 1 \end{array} \bigg| x \right)
\] (27)

**Proof.**
\[
\lim_{\omega \to 0} (\omega)_n \ _rF_s \left( \begin{array}{c} a_1(\omega), \ldots, a_r(\omega) \\ \omega, b_2(\omega), \ldots, b_s(\omega) \end{array} \bigg| x \right) = \lim_{\omega \to 0} (\omega)_n \sum_{k=0}^{\infty} \frac{(a_1(\omega))_k \times \cdots \times (a_r(\omega))_k}{\Gamma(\omega k) \Gamma(b_2(\omega) k) \cdots \Gamma(b_s(\omega) k)} x^k \Gamma(1)_k \\
= \lim_{\omega \to 0} \Gamma(\omega + n) \sum_{k=0}^{\infty} \frac{(a_1(\omega))_k \times \cdots \times (a_r(\omega))_k}{\Gamma(\omega + k) \Gamma(b_2(\omega) k) \cdots \Gamma(b_s(\omega) k)} x^k \Gamma(1)_k \\
= \Gamma(n) \sum_{k=0}^{\infty} \frac{(a_1(0))_k \times \cdots \times (a_r(0))_k}{\Gamma(k) \Gamma(b_2(0) k) \cdots \Gamma(b_s(0) k)} x^k \Gamma(1)_k \\
= \Gamma(n) \sum_{k=0}^{\infty} \frac{(a_1(0))_{k+1} \times \cdots \times (a_r(0))_{k+1}}{\Gamma(k + 1) \Gamma(b_2(0) k+1) \cdots \Gamma(b_s(0) k+1)} x^{k+1} \Gamma(1)_k +_1 \\
= \Gamma(n) \prod_{j=1}^{r} \frac{a_j(0)}{b_j(0)} \sum_{k=0}^{\infty} \frac{(a_1(0) + 1)_k \times \cdots \times (a_r(0) + 1)_k}{\Gamma(k) \Gamma(b_2(0) + 1)_k \cdots \Gamma(b_s(0) + 1)_k} x^k \Gamma(2)_k \\
= x \Gamma(n) \prod_{j=1}^{r} \frac{a_j(0)}{b_j(0)} \sum_{k=0}^{\infty} \frac{(a_1(0) + 1)_k \times \cdots \times (a_r(0) + 1)_k}{\Gamma(k) \Gamma(b_2(0) + 1)_k \cdots \Gamma(b_s(0) + 1)_k} x^k \Gamma(2)_k \\
\]
\[ = x \Gamma (n) \prod_{j=1}^{r} \frac{a_j (0)}{b_j (0)} r F_s \left( \begin{array}{c} a_1 (0) + 1, \ldots, a_r (0) + 1 \\ 2, b_2 (0) + 1, \ldots, b_s (0) + 1 \end{array} \mid x \right) \]

\[ \]

\textbf{Corollary 9} We can extend the class of Laguerre polynomials by defining

\[ L_{0}^{(-1)} = 1, \quad L_{n}^{(-1)} (x) = -\frac{x}{n} L_{n-1}^{(1)} (x), \quad n \geq 1. \quad (28) \]

\textbf{Proof.} Using the definition [36]

\[ L_{n}^{(\alpha)} (x) = \frac{(\alpha + 1)_n}{n!} \ _1 F_1 \left( \begin{array}{c} -n \\ \alpha + 1 \end{array} \mid x \right), \quad (29) \]

and (27), we get

\[ \lim_{\alpha \to -1} L_{n}^{(\alpha)} (x) = \frac{1}{n!} x \Gamma (n) (-n) \ _1 F_1 \left( \begin{array}{c} -n + 1 \\ \frac{n}{2} \end{array} \mid x \right). \]

From (29) we have

\[ \ _1 F_1 \left( \begin{array}{c} -n + 1 \\ \frac{n}{2} \end{array} \mid x \right) = \frac{(n - 1)!}{(2)_n} L_{n-1}^{(1)} (x) = \frac{1}{n} L_{n-1}^{(1)} (x), \quad n \geq 1. \]

Therefore,

\[ \lim_{\alpha \to -1} L_{n}^{(\alpha)} (x) = -\frac{x}{n} L_{n-1}^{(1)} (x), \quad n \geq 1. \]

\[ \]

\textbf{Corollary 10} We have the representation

\[ \Psi_n \left( x; a, b, \frac{b^4}{4} \right) = a \left( \frac{b}{2} \right)^{n-1} (n - 1)! L_{n-1}^{(1)} \left( -\frac{2a}{b} x \right), \quad n \geq 1. \]

\textbf{Remark 11} If we define the inner product

\[ \langle f, g \rangle_\alpha = \int_{-\infty}^{\infty} f(x) g(x) e^{-x} x^\alpha \chi_{(0, \infty)}(x) dx, \quad (30) \]
we have [36]

\[
\langle L^{(-1)}_n, L^{(-1)}_m \rangle_{-1} = \left\langle -\frac{x}{n} L^{(1)}_{n-1}, -\frac{x}{m} L^{(1)}_{m-1} \right\rangle_{-1} = \frac{1}{nm} \int_0^{\infty} L^{(1)}_{n-1}(x)L^{(1)}_{m-1}(x)x^2e^{-x}x^{-1}dx = \frac{1}{nm} \langle L^{(1)}_{n-1}, L^{(1)}_{m-1} \rangle_1
\]

\[
= \frac{1}{n^2(n-1)!} \Gamma(n+1) \delta_{n-1,m-1} = \frac{1}{n} \delta_{n,m}, \quad n, m \geq 1.
\]

Therefore, \( \{ L^{(-1)}_n \mid n \geq 1 \} \) is an orthogonal set with respect to the inner product \( \langle \cdot, \cdot \rangle_{-1} \) defined by (30). The general case was studied in [23] and [34], where it was shown that \( \{ L^{(-k)}_n \mid n \geq k \} \) is an orthogonal set with respect to the inner product \( \langle \cdot, \cdot \rangle_{-k} \) for all \( k \geq 1 \).

**Remark 12** We will now show that \( \{ L^{(-1)}_n \mid n \geq 0 \} \) is an orthonormal set with respect to the inner product

\[
\langle f, g \rangle = f(0)g(0) + \int_0^{\infty} f'(x)g'(x)e^{-x}dx.
\]

From (28) we have

\[
L^{(-1)}(0) = \delta_{n,0}, \quad n \geq 0, \quad (31)
\]

which implies

\[
\langle L^{(-1)}_0, L^{(-1)} \rangle = L^{(-1)}_n(0) + \int_0^{\infty} 0e^{-x}dx = \delta_{n,0}, \quad n \geq 0.
\]

Using the formula [40]

\[
\frac{d}{dx} L^{(\alpha)}_n = -L^{(\alpha+1)}_{n-1}, \quad n \geq 0
\]

and (31), we get

\[
\langle L^{(-1)}_n, L^{(-1)}_m \rangle = L^{(-1)}_n(0)L^{(-1)}_m(0) + \int_0^{\infty} L^{(0)}_{n-1}(x)L^{(0)}_{m-1}(x)e^{-x}dx
\]

\[
= \delta_{n-1,m-1} = \delta_{n,m}, \quad n, m \geq 1.
\]
Thus, (31) and (32) give
\[
\langle L_n^{(-1)}, L_m^{(-1)} \rangle = \delta_{n,m}, \quad n, m \geq 0.
\]

The general case for \( L_n^{(-k)} \) was first considered in [38] and subsequently in [35], [43], [48], [50], [51], [53] and [60].

3.2 Case 2: \( b^2 - 4c < 0 \)

From (19) we see that in this case \( R^\pm \) are complex conjugates,
\[
R^\pm = \frac{1}{2} \left[ b \pm i\sqrt{4c - b^2} \right]
\] (33)
with absolute value \( |R^\pm| = \sqrt{c} \). Hence, we have
\[
\frac{R^\pm}{\sqrt{c}} = e^{i\phi}, \quad 0 < \phi < \pi.
\] (34)

Using (34) in (20) we can write \( f(z) \) as
\[
f(z) = \frac{a\bar{c}}{\sqrt{4c - b^2}} \ln \left[ \frac{1 - z\sqrt{c}e^{i\phi}}{1 - z\sqrt{c}e^{-i\phi}} \right].
\] (35)

Comparing (35) and the generating function for the Meixner-Pollaczek polynomials, given by
\[
(1 - ze^{i\phi})^{-\lambda+ix} (1 - ze^{-i\phi})^{-\lambda-ix} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x; \phi) z^n, \quad \lambda > 0, \quad 0 < \phi < \pi,
\] (36)
we conclude that
\[
\Psi_n(x; a, b, c) = n!c^\frac{n}{2} \lim_{\lambda \to 0} P_n^{(\lambda)} \left( \frac{ax}{\sqrt{4c - b^2}}; \phi \right), \quad b^2 - 4c < 0
\] (37)
with \( \phi \) defined by (34). We will then find the limit in (37) using Lemma 8.

Proposition 13 The family of Meixner-Pollaczek polynomials can be extended if we define
\[
P_0^{(0)}(x; \phi) = 1, \quad P_n^{(0)}(x; \phi) = \frac{2x}{n} \sin(\phi) P_{n-1}^{(1)}(x; \phi), \quad n \geq 1.
\] (38)
Proof. Using the definition [36]

\[ P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{i\phi} \binom{\lambda + ix}{2\lambda} \left( 1 - e^{-2i\phi} \right) \]  

and (27), we get

\[
\lim_{\lambda \to 0} P_n^{(\lambda)}(x; \phi) = \frac{1}{n!} e^{i\phi} (1 - e^{-2i\phi}) \Gamma(n) (-n)(ix) \\
\times \binom{-n + 1, ix + 1}{2} \left| 1 - e^{-2i\phi} \right|
\]

\[
= \frac{1}{n!} e^{i\phi} (1 - e^{-2i\phi}) \Gamma(n) (-n)(ix) P_n^{(1)}(x; \phi) \frac{(n - 1)!}{(2)_{n-1}} e^{-i(n-1)\phi}
\]

\[
= \frac{1}{n} \left( \frac{e^{i\phi} - e^{-i\phi}}{i} \right) x P_{n-1}^{(1)}(x; \phi), \quad n \geq 1
\]

and the result follows. ■

Corollary 14 If \( b^2 - 4c < 0 \), we have the representation

\[
\Psi_n(x; a, b, c) = (n - 1)! c^{\frac{n-1}{2}} a x P_{n-1}^{(1)} \left( \frac{a}{\sqrt{4c - b^2}} x; \phi \right), \quad n \geq 1
\]  

with \( \phi \) defined by (34).

Proof. Using (38) in (37) we have

\[
\Psi_n(x; a, b, c) = n! c^{\frac{n}{2}} a \frac{2x}{n} \sin(\phi) P_{n-1}^{(1)} \left( \frac{a}{\sqrt{4c - b^2}} x; \phi \right), \quad n \geq 1
\]

From (33) and (34) we have

\[
\frac{1}{2\sqrt{c}} \left[ b \pm i\sqrt{4c - b^2} \right] = \cos(\phi) \pm i \sin(\phi)
\]

which gives

\[
\sin(\phi) = \sqrt{1 - \frac{b^2}{4c}}.
\]

\[
\boxed{}
\]

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Remark 15 Replacing $a = 1, b = 0$ and $c = 1$ in (40) and using (23), we obtain

$$T_n(x) = \Psi_n(x; 1, 0, 1) = (n - 1)! x P^{(1)}_{n-1} \left( \frac{x}{2}; \frac{\pi}{2} \right), \quad n \geq 1$$

which agrees with (3).

Remark 16 If we define the inner product

$$\langle f, g \rangle_\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)g(x)e^{(2\phi-\pi)x} |\Gamma (\lambda + ix)|^2 dx,$$  \hspace{1cm} (41)

we have [36]

$$\langle P_n^{(0)}, P_m^{(0)} \rangle_0 = \left\langle \frac{2x}{n} \sin (\phi) P^{(1)}_n (x; \phi), \frac{2x}{m} \sin (\phi) P^{(1)}_m (x; \phi) \right\rangle_0$$

$$= \frac{4 \sin^2 (\phi)}{nm} \frac{1}{2\pi} \int_{-\infty}^{\infty} P^{(1)}_{n-1} (x; \phi) P^{(1)}_{m-1} (x; \phi) e^{(2\phi-\pi)x} |\Gamma (1 + ix)|^2 x^2 dx$$

$$= \frac{4 \sin^2 (\phi)}{nm} \frac{1}{2\pi} \int_{-\infty}^{\infty} P^{(1)}_{n-1} (x; \phi) P^{(1)}_{m-1} (x; \phi) e^{(2\phi-\pi)x} |\Gamma (1 + ix)|^2 dx$$

$$= \frac{4 \sin^2 (\phi)}{nm} \frac{\Gamma (n + 1)}{4 \sin^2 (\phi) (n - 1)!} \delta_{n-1,m-1} = \frac{1}{n} \delta_{n,m}, \quad n, m \geq 1.$$

Therefore, $\{ P_n^{(0)} \mid n \geq 1 \}$ is an orthogonal set with respect to the inner product $\langle \cdot, \cdot \rangle_0$ defined by (41).

Remark 17 We will now show that $\{ P_n^{(0)} \mid n \geq 0 \}$ is an orthonormal set with respect to the inner product

$$\langle f, g \rangle = f(0)g(0) + \frac{1}{4\pi \sin (\phi)} \int_{-\infty}^{\infty} \delta f(x)\delta g(x)e^{(2\phi-\pi)x} \left| \Gamma \left( \frac{1}{2} + ix \right) \right|^2 dx,$$
where
\[ \delta f(x) = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right). \]

From (38) we have
\[ P_n^{(0)}(0; \phi) = \delta_{n,0}, \quad n \geq 0, \]
which implies
\[ \langle P_0^{(0)}, P_n^{(0)} \rangle = P_n^{(0)}(0; \phi) + \int_{-\infty}^{\infty} 0 e^{(2\phi - \pi)x} \left| \Gamma\left(\frac{1}{2} + ix\right) \right|^2 \, dx = \delta_{n,0}, \quad n \geq 0. \]  

(43)

Using the formula [36]
\[ \delta P_n^{(\lambda)}(x; \phi) = 2 \sin(\phi) P_{n-\frac{1}{2}}^{(\lambda+\frac{1}{2})}(x; \phi), \quad n \geq 0 \]
and (42), we get
\[ \langle P_n^{(0)}, P_m^{(0)} \rangle = P_n^{(0)}(0; \phi) P_m^{(0)}(0; \phi) + 2 \sin(\phi) \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{n-\frac{1}{2}}^{(1)}(x; \phi) P_{m-\frac{1}{2}}^{(1)}(x; \phi) e^{(2\phi - \pi)x} \left| \Gamma\left(\frac{1}{2} + ix\right) \right|^2 \, dx \]
\[ = 2 \sin(\phi) \frac{\Gamma(n)}{2\sin(\phi)(n-1)!} \delta_{n-1,m-1} = \delta_{n,m}, \quad n, m \geq 1. \]

Thus, (43) and (44) give
\[ \langle P_n^{(0)}, P_m^{(0)} \rangle = \delta_{n,m}, \quad n, m \geq 0. \]

3.3 Case 3: \( b^2 - 4c > 0 \)

From (19) we have \( R^- < R^+ \), if \( b^2 - 4c > 0 \). Since \( \sqrt{b^2 - 4c} < |b| \),
\[ 0 < R^- < R^+ \] for \( b > 0 \) and \( R^- < R^+ < 0 \) for \( b < 0 \).

Thus,
\[ 0 < \frac{R^-}{R^+} < 1 \] for \( b > 0 \) and \( 0 < \frac{R^+}{R^-} < 1 \) for \( b < 0 \).  

(45)
Comparing (20) and the generating function for the Meixner polynomials, given by [54]

\[
(1 - \frac{t}{\gamma})^x (1 - t)^{-\beta - x} = \sum_{n=0}^{\infty} M_n^{(\beta)}(x; \gamma) \frac{t^n}{n!}, \quad \beta > 0, \quad 0 < \gamma < 1,
\]
we have

\[
\Psi_n (x; a, b, c) = \begin{cases} 
(R^-)^n \lim_{\beta \to 0} M_n^{(\beta)} \left( -\frac{ax}{\sqrt{b^2 - 4c} - \frac{R^-}{R^+}} \right), & b > 0 \\
(R^+)^n \lim_{\beta \to 0} M_n^{(\beta)} \left( -\frac{ax}{\sqrt{b^2 - 4c} + \frac{R^+}{R^-}} \right), & b < 0 
\end{cases} 
\tag{46}
\]

where we have used (45). To find the limit in (46) we use Lemma 1.

**Proposition 18** The family of Meixner polynomials can be extended if we define

\[
M_0^{(0)}(x; \gamma) = 1, \quad M_n^{(0)}(x; \gamma) = \left( 1 - \frac{1}{\gamma} \right) xM_{n-1}^{(2)}(x - 1; \gamma), \quad n \geq 1. \tag{47}
\]

**Proof.** Using the definition [54]

\[
M_n^{(\beta)}(x; \gamma) = \binom{\beta}{n} {}_2F_1 \left( \binom{-n, -x}{\beta} \left| 1 - \frac{1}{\gamma} \right. \right)
\]

and (27), we get

\[
\lim_{\beta \to 0} M_n^{(\beta)}(x; \gamma) = \left( 1 - \frac{1}{\gamma} \right) \Gamma(n)(-n)(-x) {}_2F_1 \left( \binom{-n+1, -x+1}{2} \left| 1 - \frac{1}{\gamma} \right. \right) = \left( 1 - \frac{1}{\gamma} \right) xM_{n-1}^{(2)}(x - 1; \gamma), \quad n \geq 1
\]

and the result follows. \(\blacksquare\)

**Corollary 19** If \(b^2 - 4c > 0\), we have the representation

\[
\Psi_n (x; a, b, c) = \begin{cases} 
ax (R^-)^{n-1} M_{n-1}^{(2)} \left( -\frac{ax}{\sqrt{b^2 - 4c} - \frac{R^-}{R^+}} - 1; \frac{R^-}{R^+} \right), & b > 0 \\
ax (R^+)^{n-1} M_{n-1}^{(2)} \left( -\frac{ax}{\sqrt{b^2 - 4c} + \frac{R^+}{R^-}} - 1; \frac{R^+}{R^-} \right), & b < 0 
\end{cases} \quad n \geq 1. \tag{48}
\]
\textbf{Proof.} Using (47) in (48), we have
\[
\Psi_n(x; a, b, c) = (R^-)^n \left(1 - \frac{R^-}{R^+}\right) \left(-\frac{ax}{\sqrt{b^2 - 4c}}\right) M^{(2)}_{n-1} \left(-\frac{ax}{\sqrt{b^2 - 4c}} - 1; \frac{R^-}{R^+}\right) \\
= (R^-)^{n-1} \left(-\sqrt{b^2 - 4c}\right) \left(-\frac{ax}{\sqrt{b^2 - 4c}}\right) M^{(2)}_{n-1} \left(-\frac{ax}{\sqrt{b^2 - 4c}} - 1; \frac{R^-}{R^+}\right)
\]
for \(b > 0\) and
\[
\Psi_n(x; a, b, c) = (R^+)^n \left(1 - \frac{R^-}{R^+}\right) \frac{ax}{\sqrt{b^2 - 4c}} M^{(2)}_{n-1} \left(\frac{ax}{\sqrt{b^2 - 4c}} - 1; \frac{R^-}{R^+}\right) \\
= (R^+)^{n-1} \sqrt{b^2 - 4c} \frac{ax}{\sqrt{b^2 - 4c}} M^{(2)}_{n-1} \left(\frac{ax}{\sqrt{b^2 - 4c}} - 1; \frac{R^-}{R^+}\right)
\]
for \(b < 0.\)

\textbf{Remark 20} If we define the inner product
\[
\langle f, g \rangle_\beta = \sum_{k=0}^{\infty} f(k)g(k)\gamma^k \frac{\Gamma(\beta + k)}{k!}, \tag{49}
\]
we have [54]
\[
\langle M^{(\beta)}_n, M^{(\beta)}_m \rangle_\beta = \frac{n!\Gamma(\beta + n)}{\gamma^n (1 - \gamma)^\beta} \delta_{n,m}, \quad n, m \geq 0.
\]
Therefore,
\[
\langle M^{(0)}_n, M^{(0)}_m \rangle_0 = \left\langle \left(1 - \frac{1}{\gamma}\right) xM^{(2)}_{n-1}(x - 1; \gamma), \left(1 - \frac{1}{\gamma}\right) xM^{(2)}_{m-1}(x - 1; \gamma) \right\rangle_0 \\
= \left(1 - \frac{1}{\gamma}\right)^2 \sum_{k=1}^{\infty} k^2 M^{(2)}_{n-1}(k - 1; \gamma)M^{(2)}_{m-1}(k - 1; \gamma)\gamma^k \frac{\Gamma(k)}{k!} \\
= \left(1 - \frac{1}{\gamma}\right)^2 \sum_{k=0}^{\infty} M^{(2)}_{n-1}(k; \gamma)M^{(2)}_{m-1}(k; \gamma)\gamma^k (k + 1) \\
= \left(1 - \frac{1}{\gamma}\right)^2 \left\langle M^{(2)}_{n-1}, M^{(2)}_{m-1} \right\rangle_2 \\
= \left(1 - \frac{1}{\gamma}\right)^2 \frac{(n - 1)!\Gamma(n + 1)}{\gamma^n (1 - \gamma)^2} \delta_{n-1,m-1} = \frac{(n - 1)!n!}{\gamma^{n+1}} \delta_{n,m}, \quad n, m \geq 1.
\]
Hence, \( \{ M_n^{(0)} \mid n \geq 1 \} \) is an orthogonal set with respect to the inner product \( \langle \cdot , \cdot \rangle_0 \) defined by (49).

**Remark 21** We will now show that \( \{ M_n^{(0)} \mid n \geq 0 \} \) is an orthogonal set with respect to the inner product

\[
\langle f, g \rangle = f(0)g(0) + \sum_{k=0}^{\infty} \Delta f(k) \Delta g(k) \frac{\gamma^k}{1 - \gamma},
\]

where

\[
\Delta f(x) = f(x + 1) - f(x).
\]

From (47) we have

\[
M_n^{(0)}(0; \gamma) = \delta_{n,0}, \quad n \geq 0,
\]

which implies

\[
\langle M_0^{(0)}, M_n^{(0)} \rangle = M_n^{(0)}(0; \gamma) + \sum_{k=0}^{\infty} 0 \frac{\gamma^k}{1 - \gamma} = \delta_{n,0}, \quad n \geq 0.
\]

Using the formula [54]

\[
\Delta M_n^{(\beta)}(x; \gamma) = n \left( 1 - \frac{1}{\gamma} \right) M_{n-1}^{(\beta+1)}(x; \gamma), \quad n \geq 0
\]

and (50), we get

\[
\langle M_n^{(0)}, M_m^{(0)} \rangle = M_n^{(0)}(0; \gamma)M_m^{(0)}(0; \gamma)
\]

\[
+ \sum_{k=0}^{\infty} n \left( 1 - \frac{1}{\gamma} \right) M_{n-1}^{(1)}(k; \gamma) m \left( 1 - \frac{1}{\gamma} \right) M_{m-1}^{(1)}(k; \gamma) \frac{\gamma^k}{1 - \gamma}
\]

\[
= n^2 \left( 1 - \frac{1}{\gamma} \right)^2 \frac{(n-1)! \Gamma(n)}{\gamma^{n-1} (1 - \gamma)^2} \delta_{n-1,m-1}
\]

\[
= \frac{1}{\gamma^n} (n!)^2 \delta_{n,m}, \quad n, m \geq 1.
\]

Thus, (51) and (52) give

\[
\langle M_n^{(0)}, M_m^{(0)} \rangle = \frac{1}{\gamma^n} (n!)^2 \delta_{n,m}, \quad n, m \geq 0.
\]

Some extensions using a similar inner-product were studied in [5], [7], [8] and [21].
References


