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THE CONNECTED ISENTROPES CONJECTURE IN A SPACE OF QUARTIC POLYNOMIALS

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Abstract. This note is a shortened version of my dissertation paper, defended at Stony Brook University in December 2004. It illustrates how dynamic complexity of a system evolves under deformations. The objects I considered are quartic polynomial maps of the interval that are compositions of two logistic maps. In the parameter space P^Q of such maps, I considered the algebraic curves corresponding to the parameters for which critical orbits are periodic, and I called such curves left and right bones. Using quasiconformal surgery methods and rigidity, I showed that the bones are simple smooth arcs that join two boundary points. I also analyzed in detail, using kneading theory, how the combinatorics of the maps evolve along the bones. The behavior of the topological entropy function of the polynomials in my family is closely related to the structure of the bone-skeleton. The main conclusion of the paper is that the entropy level-sets in the parameter space that was studied are connected.

1. **Previous work and summary of results.** This paper illustrates how dynamic complexity of a system evolves under deformations. This evolution is in general only partly understood. Previous work has approached simple examples of dynamical systems from a quantitative perspective and has made use of the topological entropy $h(f)$ as a particularly useful measure of the complexity of the iterated map f. A lot has been said about entropy, although not much on monotonicity.

The logistic family $\{f_\mu(x) = \mu x(1-x)\ ,\ \mu \in [0,4]\}$ illustrates many of the important phenomena that occur in Dynamics. The theory in this case is the most complete (see [5]): $\mu \to h(f_\mu)$ is continuous, monotonically increasing, and different values $h_0 = h(f_\mu)$ are realized not only for a single μ in some cases but also for infinitely many in other cases.

The cubic polynomials on the unit interval are a 2-parameter family. In the compact parameter space of this family, the level sets of the entropy, called isentropes, were proved to be connected (see [6] and [16]). The argument makes use of a theorem of Heckman (see [8]), which does not apply in the context of higher degree polynomials.

In general, families of degree d polynomials depend on $d-1$ parameters, so the same concepts are harder to inspect for higher degrees. It is most natural to research next a family of quartic polynomials that depends only on two parameters. This paper focuses on showing the **Connected Isentropes Conjecture** for the family

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of alternate compositions of two logistic maps. More detailed proofs to some results presented here can be found in my dissertation paper [17], as cited.

I organized the ideas as follows:

I introduce the *diorbit* as an extension of the orbit of a point, to accommodate alternate iterations of two distinct (+,-) interval maps h_1 and h_2. I call *d'itinerary* the symbolic sequence extending correspondingly the classical notion of itinerary. I briefly study the diorbits of the two critical points of h_1 and h_2, respectively. For the cases when they are $2n$-periodic, I introduce a way to keep track of the succession of the diorbit points along the unit interval I by defining the *order-data* of the diorbit as a pair of permutations $(\sigma,\tau)\in S_n^2$.

My focus is on alternate iterations of two logistic maps and on the phenomena that appear in the parameter space P^Q corresponding to such pairs. For a given order-data (σ,τ), I define the left and right bones in P^Q to be the algebraic curves of parameters at which either critical point has periodic orbit of order-data (σ,τ).

To obtain combinatorial properties along the bones (such as information on their crossings and boundary points), I compare the space P^Q with a model space P^{ST} of compositions of stunted tent maps. This technique is not accidental; the stunted sawtooth maps are generally useful models in kneading-theory, because they are rich enough to encode in a canonical way all possible kneading-data of m-modal maps. My combinatorial results used Thurston's Uniqueness Theorem and an extension of it due to Poirier and interpreted by [16].

The correspondence between the two parameter spaces P^Q and P^{ST} turns out to be topologically very strong (homeo of cell complexes). To pin down the homeomorphic relationship, I need (besides combinatorics results) a few results addressing the geometry and the degree of smoothness of the bones in P^Q. The geometric results require crucial use of a rigidity theorem to prove density of hyperbolicity in P^Q ([9]). To prove $\mathcal{C}^1$ smoothness, I perform a quasiconformal surgery construction to perturb a map with a superattracting cycle to a map having an attracting cycle with small nonzero multiplier.

Finally, I emphasize the relations between the behavior of the entropy in P^Q and the bone structure. The main result, connectedness of the entropy level-sels, is obtained by topological correspondence with the level-sets in the model space P^{ST}, which are contractible.

2. **Combinatorics.**

2.1. **A discussion on the kneading-data.** Let $h : I \to I$ be an *m-modal map* of the interval, i.e., there exist $0 < \mathbf{c_1} \le \mathbf{c_2} \le ... \le \mathbf{c_m} < 1$ *folding points* or *critical points* of h such that h is alternately increasing and decreasing on the intervals $H_0, ..., H_m$ between the folding points.

$$I = \bigcup_{k=0}^{m} H_k \cup \bigcup_{j=1}^{m} \{\mathbf{c_j}\}.$$

We say that h is of shape s=(+,-,+,...) if h is increasing on H_0 and of shape s=(-,+,-,...) if h is decreasing on H_0. We say that h is *strictly m-modal* if there is no smaller m with the properties above.

We define the *itinerary* $\Im(x) = (A_0(x), A_1(x), ...)$ of a point $x \in I$ under h as a sequence of symbols in $\mathcal{A} = \{H_0, ..., H_m\} \cup \{\mathbf{c_1}, ..., \mathbf{c_m}\}$, where

$$\begin{cases} A_k(x) = H_j, & \text{if} \quad f^{\circ k}(x) \in H_j \\ A_k(x) = \mathbf{c_j}, & \text{if} \quad f^{\circ k}(x) = \mathbf{c_j}. \end{cases}$$

The *kneading sequences* of the map h are defined as the itineraries of its folding values:

$$\mathcal{K}_j = \mathcal{K}(\mathbf{c_j}) = \Im(f(\mathbf{c_j})),\ 1 \leq j \leq m.$$

The *kneading-data* $\mathbf{K}$ of h is the m-tuple of kneading-sequences:

$$\mathbf{K} = (\mathcal{K}_1, ..., \mathcal{K}_m).$$

The simplest example of an m-modal map is a *sawtooth map* with m teeth (see Figure 1.1(a)).

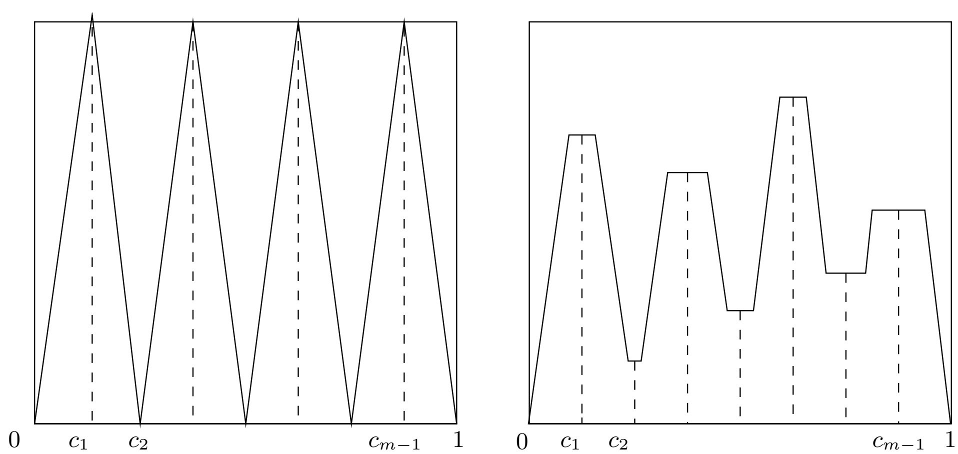


FIGURE 1. *(a)Sawtooth map of the interval. (b)Stunted sawtooth map*

We call a *stunted sawtooth map* a sawtooth map whose peaks have been stunted by plateaus placed at chosen heights (see Figure 1.1(b)). Its folding points are considered to be the centers of the plateaus. For this paper's specific purposes we will need to look at tent maps (1-modal sawtooth maps) and their stunted version, which we will call stunted tent maps.

Another simple and rich example of m-modal maps is the collection of $(m+1)$-degree polynomials from I to itself. The folding points could be taken in this case to be the critical points of the polynomial (in the classical sense) of odd order. In the context of polynomial m-modal maps, we have a powerful tool to use in *Thurston's Uniqueness Theorem.*

Definition 2.1.1. A polynomial map is called *post-critically finite* if the orbit of every critical point is periodic or eventually periodic.

Theorem 2.1.1. ***Thurston Uniqueness Theorem for Real Polynomial Maps:*** *A post-critically finite real polynomial map of degree m+1 with m distinct real critical points is uniquely determined, up to a positive affine conjugation, by its kneading data.*

We will also use a converse of this basic theorem of Thurston, due to Poirier (as interpreted by [16]).

Definition 2.1.2. We say that a symbolic sequence $\Im(x) = (A_0(x), A_1(x), ...)$ is *flabby* if some point of the associated orbit, which is not a folding point, and an immediately adjacent folding point have corresponding folding values with identical itineraries. A symbolic sequence is called *tight* if it is not flabby. The kneading data of a map is tight if each of its kneading sequences is tight.

Lemma 2.1.1. *The kneading data of a stunted sawtooth map is tight if and only if the orbit of each folding point never hits a plateau except at its folding point.*

Theorem 2.1.2. *Suppose that the kneading data* $\mathbf{K}$ *is achieved by some* m*-modal map of shape* s*, with* $\mathcal{K}_i \neq \mathcal{K}_j$ *for all* i*. Then there exists a post-critically finite polynomial map of degree* $m+1$ *and shape* s *with kneading-data* $\mathbf{K}$ *if and only if each* $\mathcal{K}_i$ *is periodic or eventually periodic, and also tight. This polynomial is always unique when it exists, up to a positive affine change of coordinates.*

2.2. **Definitions and first goals.**

In light of the general definition given in Section 2.1, we call a *boundary anchored, (+,-) unimodal map* of the unit interval a map $h\colon I = [0,1] \to I$ such that $h(0) = h(1) = 0$ and such that there exists $\gamma \in (0,1)$, called *folding* or *critical* point, with h increasing on $(0, \gamma)$ and decreasing on $(\gamma, 1)$. The orbit of a point $x \in I$ under such an h will be the sequence of iterates $\{h^{\circ n}(x)\}_{n \geq 0}$. The itinerary of x under h is the sequence $(J_0, J_1, ...)$ of symbols L (left), R (right) and Γ (center, or critical) such that:

$$\begin{cases} J_j = L, & \text{if } h^{\circ j}(x) < \gamma \\ J_j = R, & \text{if } h^{\circ j}(x) > \gamma \\ J_j = \Gamma, & \text{if } h^{\circ j}(x) = \gamma. \end{cases}$$

The next few sections of this paper are dedicated to the study of the combinatorics of the dynamical system we are considering: generated by alternate iterates of two unimodal interval maps. In this sense, it is convenient to consider two copies of the unit interval $I_1 = I_2 = I$ and think of our pair of maps (h_1, h_2) as a self map of the disjoint union $I_1 \sqcup I_2 \to I_1 \sqcup I_2$, which carries I_1 to I_2 as h_1 and I_2 to I_1 as h_2, with critical points $\gamma_1 \in I_1$ and $\gamma_2 \in I_2$, respectively.

We call an *diorbit* under the pair (h_1, h_2) a sequence:

$$x \to h_1(x) \to h_2(h_1(x)) \to h_1(h_2(h_1(x)))...$$

We say a diorbit is *critical* if it contains either critical point γ_1 or γ_2. A critical diorbit that contains both γ_1 and γ_2 will be called *bicritical*.

We call the *d'itinerary* of a point x under (h_1, h_2) the infinite sequence $\Im(x) = \{J_k(x)\}_{k \geq 0}$ of alternating symbols in $\{L_1, \Gamma_1, R_1\}$ and $\{L_2, \Gamma_2, R_2\}$ that expresses the positions of the iterates of x in I_1 and I_2 with respect to γ_1 or γ_2.

Clearly, not all arbitrary sequences of appropriate symbols are in general admissible as d'itineraries of a point under a pair of given maps.

It is fairly easy to show that the pair of critical d'itineraries $(\Im(\gamma_1), \Im(\gamma_2))$ under a pair (h_1, h_2) of unimodal maps determines the kneading-data of $h_2 \circ h_1$ and conversely (the proof is left as an exercise). In particular this applies to pairs of stunted tent maps and to pairs of logistic maps, which are the objects of this paper.

We will use the same order on admissible d'itineraries as the total order used on regular itineraries (see [3]), which is consistent with the order of points on the real line:

$$\Im(x) < \Im(x') \Rightarrow x < x'$$
$$x < x' \Rightarrow \Im(x) \leq \Im(x').$$

We say that the diorbit of x is *periodic* of period $2n$ under (h_1, h_2) if n is the smallest positive integer such that $(h_2 \circ h_1)^{\circ n}(x) = x$ (i.e., x has period n under the composition $(h_2 \circ h_1)$). We will use the following notation for a $2n$-periodic diorbit under (h_1, h_2):

$$x_1 = x_{i_1} \xrightarrow{h_1} y_{j_1} \xrightarrow{h_2} x_{i_2} \xrightarrow{h_1} ... \xrightarrow{h_1} y_{j_n} \xrightarrow{h_2} x_{i_1}, \qquad (1)$$

where $\{x_i\}_{1\leq i\leq n} \subset I_1$ and $\{y_j\}_{1\leq j\leq n} \subset I_2$ are both increasing.

Definition 2.2.1. The *order-data* of the periodic diorbit (1) is defined as the pair (σ, τ) of permutations in S_n given by

$$h_1(x_i) = y_{\sigma_i}$$
$$h_2(y_j) = x_{\tau_j}\ ,$$

so that $\sigma_{i_k} = j_k$ and $\tau_{j_k} = i_{k+1}$. (Here the subscripts must be understood as integers mod n, e.g., $i_{n+1} = i_1 = 1$.) An *admissible order-data* is a $(\sigma, \tau) \in S_n^2$ which is achieved as order-data of a periodic orbit of some pair (h_1, h_2) of interval unimodal maps.

The (+,-) unimodal shape of h_1 and h_2 imposes a set of necessary and sufficient conditions for a (σ, τ) to be admissible:

$$(I) \begin{cases} \text{If } \sigma_{i+1} < \sigma_i\ , \quad \text{then } \sigma_{j+1} < \sigma_j, \forall j \geq i \quad \text{(i.e., } \sigma \text{ monotone or unimodal)} \\ \text{If } \tau_{i+1} < \tau_i\ , \quad \text{then } \tau_{j+1} < \tau_j, \forall j \geq i \quad \text{(i.e., } \tau \text{ monotone or unimodal)} \end{cases}$$

(II) $\tau \circ \sigma$ is a cyclic permutation (i.e., has no smaller cycles).

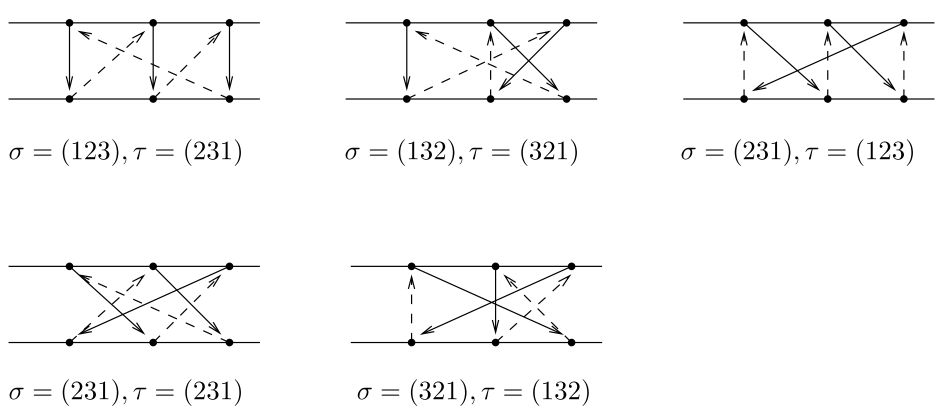


FIGURE 2. *All admissible order-data $(\sigma, \tau)$of period $2n = 6$. Each sketch represents the interval $I = I_1$ on top, with the diorbit points $x_1 < x_2 < x_3$ and the interval $I = I_2$ underneath, with the diorbit points $y_1 < y_2 < y_3$.*

The first goal will be to research the relation between the d'itinerary and the order-data of a periodic critical diorbit.

Suppose γ_1 is periodic of period $2n$ under (h_1, h_2), and let $x_1 = x_{i_1} \to y_{j_1} \to \dots \to y_{j_n} \to x_{i_1}$ be its diorbit. Then the order-data $(\sigma, \tau) \in S_n^2$ of the diorbit determines its d'itinerary via the position of the element $y_{j_l} \in I_2$ closest to γ_2. In other words, there are at most two critical d'itineraries containing γ_1 corresponding to a given order-data, and the same for γ_2. If in particular the diorbit is bicritical, then $y_{j_l} = \gamma_2$ and the d'itinerary is completely defined.

Note also that the order of points in a critical periodic diorbit of a (+,-) unimodal map is *strictly* preserved in the order of their d'itineraries, i.e., $x < x'$ implies $\Im(x) < \Im(x')$. Hence conversely, knowing the d'itinerary $\Im$ of the bicritical diorbit, we can obtain the order of occurrence of the diorbit points in I_1 and I_2, respectively. This proves the following:

Theorem 2.2.1. *If the diorbit of γ_1 is bicritical of period $2n$ under a pair of (+,-) unimodal maps (h_1, h_2), then the d'itinerary of γ_1 determines the order-data of the diorbit and conversely.*

2.3. **Parameter spaces.**

We plan to study in more detail the dynamics of our particular family of such pairs of interval unimodal maps.

Recall that the logistic map (with critical value v) is defined as $q_v(x) = 4vx(1-x)$, $x \in \mathbb{R}$. Clearly $q_v(0) = q_v(1) = 0$, for any value of the parameter v. Moreover, for $v \in [0,1]$, q_v carries the unit interval to itself, so it is a boundary anchored, (+,-) unimodal interval map.

Our goal is to study the dynamics of pairs (q_v, q_w) (i.e., of alternate iterations of q_v and q_w), where $(v,w) \in [0,1]^2$. We call the family of (q_v, q_w) the Q-family, and we parameterize it by the pair of critical values, so that the parameter space will be

$$P^Q = \{(v,w) \in [0,1] \times [0,1]\} = [0,1]^2.$$

The behavior of the pairs in the Q-family is not very well-understood. We will compare it to the dynamics in a model family much easier to research, the family of pairs (st_v, st_w) of stunted tent maps:

$$st_v : I_1 \to I_2,\ \gamma_1 = \frac{1}{2} \quad \text{and} \quad st_w : I_2 \to I_1,\ \gamma_2 = \frac{1}{2},$$

where

$$st_v(x) = \begin{cases} 2x & \text{if } x \le \frac{v}{2} \\ v & \text{if } \frac{v}{2} \le x \le 1 - \frac{v}{2} \\ 2 - 2x & \text{if } x \ge 1 - \frac{v}{2}. \end{cases}$$

Recall from 2.1 that the folding point of such a stunted tent map is by convention the midpoint $\gamma = \frac{1}{2}$, hence the critical value is $st(\gamma) = v$. We call the family of pairs of such maps the ST-family. Its corresponding parameter space will be denoted by

$$P^{ST} = \{(v,w) \in [0,1] \times [0,1]\}.$$

We aim to obtain combinatorics results in $P^Q = [0,1]^2$. However, we start by proving similar results in the parameter space $P^{ST} = [0,1]^2$ of "approximating" stunted tent maps. Comparison of the two spaces will be a frequent strategy. The topological correspondence of P^Q and P^{ST} will be eventually sustained with a rigorous proof and will enable us to translate topological properties from one space to the other.

2.4. **The combinatorics in the ST-family.**

Next, let's see how the combinatorics in Section 2.2 applies to the model ST-family:

Theorem 2.4.1. *Given $(\sigma, \tau) \in S_n^2$ admissible order-data, there is a unique pair of stunted tent maps (st_v, st_w) with periodic bicritical diorbit of order-data (σ, τ).*

Proof. First, let $\Im$ be a sequence of alternating symbols in $\{L_1, \Gamma_1, R_1\}$ and $\{L_2, \Gamma_2, R_2\}$, admissible as a bicritical d'itinerary of period $2n$ under a pair of unimodal maps:

$$\Im = (J_0 = \Gamma_1, J_1, J_2, ..., J_{2l}, J_{2l+1} = \Gamma_2, J_{2l+2}, ..., J_{2n-1}, J_{2n} = \Gamma_1, ...),$$

where $J_{2n+k} = J_k$ for all k and $J_k \neq \Gamma_1, \Gamma_2$, for all k not equal to 0 or $2l+1$ mod $2n$. There exists a unique pair of stunted tent maps (st_v, st_w) that has a bicritical diorbit of period $2n$:

$$x_1 = x_{i_1} \xrightarrow{st_v} y_{j_1} \xrightarrow{st_w} ...y_{j_n} \xrightarrow{st_v} x_{i_1} = x_1,$$

having $\Im$ as its d'itinerary.

To prove the existence, it is easier to consider a diorbit that has the required d'itinerary through a pair of regular tent maps, then stunt the maps at the highest values of the diorbit in I_1 and I_2, respectively. The proof of the uniqueness is an easy exercise (see [17]): starting with the critical points γ_1 and γ_2, iterate backwards following the branch indicated by the d'itinerary $\Im$. This way, we obtain the values of v and w.

Going back to the proof of our theorem: given an admissible order-data $(\sigma, \tau) \in S_n^2$ for a required bicritical diorbit, we can determine the d'itinerary $\Im$ of the diorbit (by Theorem 2.2.2). As shown above, we can find a unique pair (st_v, st_w) of stunted tent maps with a bicritical diorbit of length $2n$ and d'itinerary $\Im$. By the converse in Theorem 2.2.2, the order-data for the diorbit we have found will be (σ, τ). □

To make the discussion again a step more general, we return to pairs (h_1, h_2) of arbitrary unimodal maps. If both critical points γ_1 and γ_2 are periodic, then there are two possible cases that can occur: (h_1, h_2) has either a bicritical diorbit (discussed earlier) or two disjoint critical diorbits.

Definition 2.4.1. Let $(\sigma, \tau) \in S_{m+n}^2$ be a pair of permutations decomposable into two cycles: $(\sigma_1, \tau_1) \in S_m^2$ and $(\sigma_2, \tau_2) \in S_n^2$. We say that two disjoint periodic diorbits o_1 and o_2 under a pair (h_1, h_2) of (+,-) unimodal maps have *joint order-data* (σ, τ) if:

1. *o_1 has order-data (σ_1, τ_1) and o_2 has order-data (σ_2, τ_2);*
2. *the order of the points in I_1 and I_2 is given by $(\tau \circ \sigma)$ and $(\sigma \circ \tau)$ respectively (see* [16] *for the definition of* order-type*).*

We will say about a permutation $(\sigma, \tau) \in S^2_{m+n}$ that it is *admissible as a joint order-data* if there exist two disjoint diorbits under some pair of (+,-) unimodal maps which have joint order-data (σ, τ).

Similarly as for regular order-data, one can obtain the following two results (see [17] for detailed proofs):

Theorem 2.4.2. *Suppose a pair (h_1, h_2) of (+,-) unimodal maps has disjoint critical diorbits o_1 and o_2. Then their d'itineraries determine their joint order-data and conversely.*

Theorem 2.4.3. *Given $(\sigma, \tau) = ((\sigma_1, \tau_1), (\sigma_2, \tau_2)) \in S^2_{m+n}$ admissible joint order-data, there exists a unique pair (st_v, st_w) of stunted tent maps with disjoint critical diorbits $o_1 \ni \gamma_1$ and $o_2 \ni \gamma_2$ having joint order-data (σ, τ).*

2.5. Description of bones in the ST-family.

Definition 2.5.1. Fix an admissible order-data $(\sigma, \tau) \in S^2_n$. By the *left bone* in the parameter space of the ST-family we mean the set of pairs $(v, w) \in P^{ST}$ such that the critical point $\gamma_1 \in I_1$ has under (st_v, st_w) a periodic diorbit of period $2n$ and order-data (σ, τ).

We will use the notation $B^{ST}_L(\sigma, \tau)$, or B^{ST}_L if there is no ambiguity. We define a *right bone* symmetrically (i.e., we require γ_2 to be periodic of specified period and order-data), and we denote it by $B^{ST}_R(\sigma, \tau)$, or B^{ST}_R.

We will need for later a comprehensive approach to the left and right bones and their properties. Recall (from Theorem 2.4.1) that *there is a unique pair $(v_0, w_0) \in B^{ST}_L$ such that the periodic of γ_1 is bicritical (i.e., hits γ_2) under (st_{v_0}, st_{w_0}).*

Theorem 2.5.1. *For each admissible order-data (σ, τ), let (v_0, w_0) be the parameter pair for the associated bicritical diorbit in the ST-family. Then there are unique numbers $v_1 < v_0 < v_2$ so that the left bone $B^{ST}_L(\sigma, \tau)$ is the union $\{v_1, v_2\} \times [w_0, 1] \cup (v_1, v_2) \times \{w_0\}$ of three line segments, as illustrated in Figures 3 and 4. The description of the right bone $B^{ST}_R(\sigma, \tau)$ is completely analogous.*

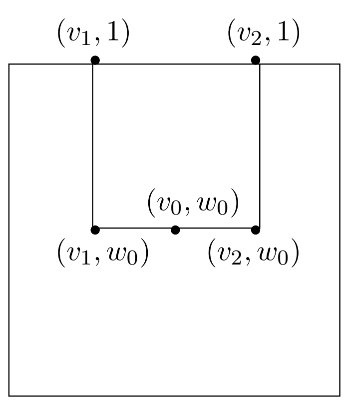


FIGURE 3. $B^{ST}_L = \{v_1, v_2\} \times [w_0, 1] \cup (v_1, v_2) \times \{w_0\} \ni (v_0, w_0)$

Proof. We can determine the shape of B_L^{ST}, hence proving 2.5.2, by constructive means, starting with the point (v_0, w_0). Under (st_{v_0}, st_{w_0}), the critical point $\gamma_1 \to v_0 \to ... \to \gamma_2 \to w_0 \to ... \to \gamma_1$. The bicritical diorbit hits each plateau only once, at its center. By sliding the first plateau up and down, the diorbit of γ_1 will change in a continuous way. For a fixed height v of the left plateau, call $y_l(v)$ the element in the diorbit of γ_1 under (st_v, st_{w_0}) that is closest to γ_2 in I_2. Clearly, if $v = v_0$, then $y_l(v_0) = \gamma_2$. We can move v continuously within an interval $[v_1, v_2] = [v_0 - \epsilon, v_0 + \epsilon]$, $\epsilon > 0$ such that $y_l(v)$ moves from $\frac{w_0}{2}$ to $1 - \frac{w_0}{2}$. Along the process, the diorbit stays periodic and the order of the occurrence of points remains consistent with (σ, τ).

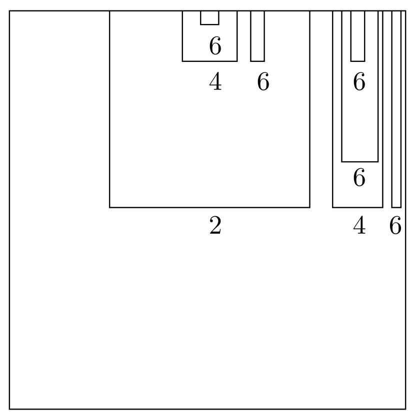


Figure 4. *Left bones in the ST-family of period at most 6.We marked by (2) the unique bone of period 2, corresponding to order-data in* $(\sigma = (1), \tau = (1)) \in S_1^2$*. (4) are the two bones of period 4 and have the two possible order-data in* S_3^2*:* $(\sigma = (12), \tau = (1)(2))$ *or* $(\sigma = (1)(2), \tau = (12)) \in S_2^2$*. (6) are the bones of period 6 and any admissible order-data:* $(\sigma = (123), \tau = (231)$*,* $(\sigma = (132), \tau = (321)), (\sigma = (231), \tau = (231)), (\sigma = (321), \tau = (132))$ *or* $(\sigma = (231), \tau = (123))$

It is not hard to see that $B_L^{ST} = \{v_1, v_2\} \times [w_0, 1] \cup (v_1, v_2) \times \{w_0\}$. In particular, there are exactly two values $v = v_1$ and $v = v_2$ such that the diorbit of γ_1 has given order-data (σ, τ) under (st_v, st_1) (i.e., there are exactly two boundary points of B_L^{ST}, on $[0, 1] \times \{1\}$). □

2.6. **Important points on the bones.**

In either space, we consider the $2n$-bones for an admissible $(\sigma, \tau) \in S_n^2$.

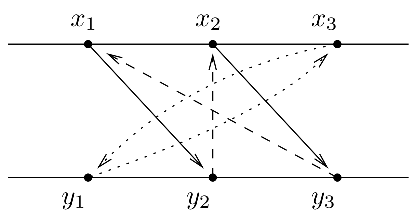
Definition 2.6.1. The *left bone* $B_L(\sigma, \tau)$ is the set of all parameters for which γ_1 has periodic diorbit of order-data (σ, τ). The *right bone* $B_R(\sigma, \tau)$ is the set of all parameters for which γ_2 has periodic diorbit of order-data (σ, τ).

We will distinguish between the two parameter spaces whenever necessary, by using the ST superscript for the bones in P^{ST} (i.e., B_L^{ST}, B_R^{ST}) and the Q superscript for the bones in P^Q (i.e., B_L^Q, B_R^Q).

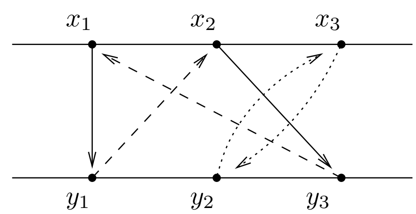
Remarks: (1) In either parameter space, any two left bones are disjoint and any two right bones are disjoint by definition.

(2) It follows easily from Theorem 2.5.2 that in the ST-family a left bone can cross a right bone only at 0, 2 or 4 points (see Figure 10).

Definition 2.6.2. In either parameter space, an intersection of $B_L(\sigma_1, \tau_1)$ and $B_R(\sigma_2, \tau_2)$ is called a *primary intersection* if $(\sigma_1, \tau_1) = (\sigma_2, \tau_2)$ and the pair of maps corresponding to the intersection has a bicritical diorbit with this order-data. It is called a *secondary intersection* if the corresponding map has disjoint critical diorbits, of order-data (σ_1, τ_1) and (σ_2, τ_2) respectively, and joint order-data (σ, τ). A *capture point* on $B_L(\sigma_1, \tau_1)$ in either P^{ST} or P^Q is a pair of maps for which γ_2 eventually maps on γ_1 such that it has an eventually periodic, but not periodic, diorbit. We define symmetrically a capture point on $B_R(\sigma_2, \tau_2)$.



$(\sigma, \tau) = ((231), (321))$



$(\sigma, \tau) = ((132), (231))$

FIGURE 5. *Combinatorics of the two secondary intersections of a period 4 left bone with a period 2 right bone. The two possible order-data are:* $(\sigma, \tau) = ((231), (321))$ *and* $(\sigma, \tau) = ((132), (231))$.

2.7. **More on kneading-data.**

In this section we will construct a bijective correspondence between bones intersections in our two parameter spaces P^{ST} and P^Q. For the proof, it is necessary to view the composition $q_w \circ q_v$ of two logistic maps, depending on the case, either as a 3-modal map with three critical points in $I = I_1$: $\mathbf{c_1} \leq \mathbf{c_2} \leq \mathbf{c_3}$ (with $\mathbf{c_2} = \gamma_1$ and $q_v(\mathbf{c_1}) = q_v(\mathbf{c_3}) = \gamma_2$) or as a unimodal map with folding point γ_1 (in case $q_v(x) = \gamma_2$ has a double real root or two complex roots).

We will use rigidity theorems that involve essentially properties of the kneading-data. So let us look in more detail at the possible kneading-data of the maps in P^{ST} and P^Q.

Maps in P^{ST}: For any $(v, w) \in P^{ST}$, the map $st_w \circ st_v$ could be considered 3-modal, with folding points $c_1 = \frac{1}{4}, c_2 = \gamma_1 = \frac{1}{2}$ and $c_3 = \frac{3}{4}$.

$$\mathcal{A}^{ST} = \{[0, \frac{1}{4}), \frac{1}{4}, (\frac{1}{4}, \frac{1}{2}), \frac{1}{2}, (\frac{1}{2}, \frac{3}{4}), \frac{3}{4}, (\frac{3}{4}, 1]\}$$

and

$$\mathbf{K}_{ST} = (\mathcal{K}(c_1), \mathcal{K}(c_2), \mathcal{K}(c_3)).$$

We can consider P^{ST} as made of three parts: $P^{ST} = P_1^{ST} \cup P_2^{ST} \cup P_3^{ST}$, where $P_1^{ST} = \{(v,w) \in [0,1]^2,\ w \geq 2v\}$, $P_2^{ST} = \{(v,w),\ w < 2v,\ w \leq 2-2v\}$, and $P_3^{ST} = \{(v,w),\ w > 2-2v\}$.

I. There are no right bones in P_1^{ST}, and hence no bones intersections.

II. P_2^{ST} contains no secondary intersections, since $\frac{w}{2} \leq v \leq 1-\frac{w}{2}$. Indeed, $st_w(\gamma_2) = w = (st_w \circ st_v)(\gamma_1)$, so the diorbits of γ_1 and γ_2 are not disjoint. Moreover, if $(v,w) \in P_2^{ST}$ is a primary intersection, then the map $st_v \circ st_w$ is strictly 3-modal, with only one exception: $(v,w) = (\frac{1}{2}, \frac{1}{2})$.

III. For $(v,w) \in P_3^{ST}$ we have that $st_w \circ st_v$ is strictly 3-modal, hence $\mathcal{K}(c_1) = \mathcal{K}(c_3) \neq \mathcal{K}(c_2)$.

Maps in P^Q: The behavior of the degree 4 polynomials in the Q-family is also different for distinct values of the parameters.

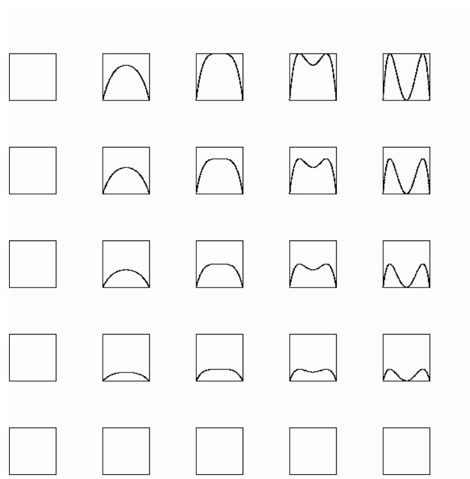


FIGURE 6. *A few examples of behavior of maps in P^Q. The critical points of the quartic map $q_w \circ q_v$ are distinct and real for $v > \frac{1}{2}$, and all coincide for $v = \frac{1}{2}$, while two of them are complex for $v < \frac{1}{2}$.*

I. If $v < \frac{1}{2}$, then $q_w \circ q_v$ has only one real critical point $C_2 = \gamma_1 = \frac{1}{2}$ and two complex ones $C_1, C_3 \in \mathbb{C} \backslash \mathbb{R}$. This parameter subset will be of somewhat less interest, as it is not crossed by any right bones and hence contains no bones intersections. Indeed, $q_v(x) \leq v < \frac{1}{2}$, $\forall x \in I_1$, so no diorbit can go through γ_2.

II. If $v = \frac{1}{2}$, then $q_w \circ q_v$ has a degenerate real critical point $C_1 = C_2 = C_3 = \gamma_1$. This line contains primary intersections with right bones. More precisely, if a left bone hits $\{v = \frac{1}{2}\}$, then the crossing point is its primary intersection. However, in this case its reflection $q_v \circ q_w$ is strictly 3-modal, with the exception of $v = w = \frac{1}{2}$, which is the period 2 primary intersection.

III. If $v > \frac{1}{2}$, then there are three distinct real critical points for $q_w \circ q_v$: $C_1 < C_2 < C_3$, with $C_2 = \gamma_1$ and $q_v(C_1) = q_v(C_3) = \gamma_2$. The map is 3-modal:

$$\mathcal{K}^Q = \{[0, C_1), C_1, (C_1, C_2), C_2, (C_2, C_3), C_3, (C_3, 1]\}$$

and

$$\mathbf{K}_Q = (\mathcal{K}(C_1), \mathcal{K}(C_2), \mathcal{K}(C_3)).$$

Remarks.(1) There are no bones in P^Q in the region $\{v < \frac{1}{2},\, w < \frac{1}{2}\}$.
(2) If $v < \frac{1}{2}$, then $q_w \circ q_v$ has complex critical points. However, if (v, w) is on a bone in the region $\{v < \frac{1}{2}\}$, then it must be on a left bone, so that $w \geq \frac{1}{2}$. Hence in this case the map $q_v \circ q_w$ corresponding to its symmetric point (w, v) on the corresponding right bone has real critical points, non-degenerate if $w \neq \frac{1}{2}$.

A correspondence is already apparent between the shape and position of two left bones with identical order-data in the two spaces P^{ST} and P^Q. For instance, the unique primary intersection of period two: $(v, w) = (\frac{1}{2}, \frac{1}{2}) \in P^{ST}$ clearly corresponds combinatorially to the point $(v, w) = (\frac{1}{2}, \frac{1}{2}) \in P^Q$. We will consider at least this case classified in our future analysis. The following theorems will therefore concern specifically the strictly 3-modal case (applicable either for $st_w \circ st_v$ and $q_w \circ q_v$ or for their reflections $st_v \circ st_w$ and $q_v \circ q_w$) (as shown by the previous remark).

2.8. **The correspondence of the bones intersections.**

We use Thurston's Theorem and its extension for boundary anchored polynomials of degree four and shape (+,-,+,-) to construct in this section a bijection between bones crossings in the two parameter spaces. For the rest of Section 2, we will need to adapt our notation to distinguish between parameters $(v, w) \in P^{ST}$ and parameters $(v', w') \in P^Q$, in order to avoid confusion.

Theorem 2.8.1. *Let $(\sigma, \tau) \in S_n^2$ be admissible order-data. There is a unique primary intersection (v', w') in P^Q with this data and conversely.*

Proof. **Uniqueness**: Suppose we have a pair $(v, w) \in P^Q$ with a bicritical diorbit of order-data (σ, τ). We implicitly know the d'itinerary of the bicritical diorbit, hence the kneading sequences of the three real distinct critical points $C_1 < C_2 = \frac{1}{2} < C_3$ of $q_w \circ q_v$ (if $v > \frac{1}{2}$) or $q_v \circ q_w$ (if $w > \frac{1}{2}$). By Thurston's Theorem, the boundary anchored polynomial of degree 4 with the expected kneading data is unique, implying the uniqueness of the pair (q_v, q_w) with the given order-data.

Existence: Let (st_v, st_w) be the pair of stunted tent maps with bicritical diorbit of order-data (σ, τ). We know by Theorem 2.2.2 that we can determine the d'itinerary of this bicritical diorbit. If we exclude the case $v = w = \frac{1}{2}$, which is already classified, then either $st_w \circ st_v$ or $st_v \circ st_w$ is strictly 3-modal (say $st_w \circ st_v$, to fix our ideas). We know the kneading-data $\mathbf{K}_{ST} = (\mathcal{K}(c_1), \mathcal{K}(c_2), \mathcal{K}(c_3))$ for $st_w \circ st_v$, which should also be the kneading-data for the polynomial $q_{w'} \circ q_{v'}$ that we want to find. We hence need to prove existence of a polynomial of degree 4 with the required kneading-data $\mathbf{K}$ and then show that it can be written as a composition of two logistic maps $q_{v'}$ and $q_{w'}$. We will finally show that the pair $(q_{v'}, q_{w'})$ we found has indeed the given order-data.

Each two consecutive kneading-sequences of $\mathbf{K}$ are distinct. Also, each $\mathcal{K}(c_i)$ hits each plateau of $st_w \circ st_v$ at most once, above its corresponding critical point. So, by Lemma 2.1.4, all kneading sequences of $\mathbf{K}$ are tight.

By Thurston's Theorem, these imply existence and uniqueness of a polynomial P with kneading-data $\mathbf{K}$, of shape (+,-,+,-) and conditions at the boundary $P(0) = 0$ and $P(1) = 0$. In general, we know that a boundary anchored polynomial P of degree 4, shape (+,-,+,-) and real distinct critical points $0 < C_1 < C_2 < C_3 < 1$ is a composition of logistic maps if and only if $P(C_1) = P(C_3)$. In our case, we know that the kneading sequences $\mathcal{K}(C_1) = \mathcal{K}(c_1)$ and $\mathcal{K}(C_3) = \mathcal{K}(c_3)$ are identical. Suppose $P(C_1) < P(C_3)$. Then the whole interval $[P(C_1), P(C_3)]$ will have the same (bicritical) d'itinerary, as $\mathcal{K}(C_1) = \mathcal{K}(C_3)$, so after a finite number of iterations under P it will all map to C_2, a contradiction. So $P(C_1) = P(C_3)$, and hence there exists a pair of quadratic maps such that $P = q_{w'} \circ q_{v'}$.

The kneading data $\mathbf{K}$ determines the d'itinerary of the bicritical diorbit and its order-data. So, the polynomial map we found can only have the given order-data (σ, τ). □

The proof is similar for the equivalent statement on secondary intersections:

Theorem 2.8.2. *Let $(\sigma, \tau) \in S^2_{m+n}$ be admissible joint order-data. There is a unique secondary intersection in P^Q with this data and conversely.*

2.9. **The correspondence of the boundary points.**

Fix $(\sigma_1, \tau_1) \in S^2_n$. The left bone $B^{ST} = B^{ST}_L(\sigma_1, \tau_1)$ in P^{ST} with order-data (σ_1, τ_1) is as an algebraic curve in P^{ST}. Its boundary consists of two points, as shown in Theorem 2.5.2:

$$\delta B^{ST} = B^{ST} \cap (I \times \{1\}) = \{(v_1, 1), (v_2, 1)\},$$

with $v_1 < v_2$.

We will use a subscript notation to distinguish between the two parameter spaces. For any $(v, w) \in P^{ST}$, we will call $\Im_{ST}(x)(v, w)$ the d'itinerary of x under (st_v, st_w) and $\mathbf{K}_{ST}(v, w)$ the kneading-data of $st_w \circ st_v$. For any $(v', w') \in P^Q$, we will call $\Im_Q(x)(v', w')$ the d'itinerary of x under $(q_{v'}, q_{w'})$ and $\mathbf{K}_Q(v', w')$ the kneading-data of $q_{w'} \circ q_{v'}$.

The d'itineraries of the critical points γ_1 and γ_2 under (st_{v_1}, st_1) and (st_{v_2}, st_1) are as follows:

$$\Im_{ST}(\gamma_1)(v_1, 1) \neq \Im_{ST}(\gamma_1)(v_2, 1)$$

$$\Im_{ST}(\gamma_2)(v_1, 1) = \Im_{ST}(\gamma_2)(v_2, 1) = (\Gamma_2, R_1, L_2, L_1, L_2, L_1, ...).$$

At any (v, w) on the left bone B^{ST}, γ_1 has a periodic diorbit o_1 of period $2n$ and order-data (σ_1, τ_1). At the two boundary points $(v_1, 1), (v_2, 1) \in \delta B^{ST}$, the diorbit o_2 of γ_2 is also finite, although not periodic. So the stunted sawtooth map $st_w \circ st_v$ is postcritically finite.

We expect the boundary of the corresponding quadratic left bone $B^Q = B^Q(\sigma_1, \tau_1)$ to look similarly to δB^{ST}:

Theorem 2.9.1. *The boundary of $B^Q(\sigma_1, \tau_1) = B^Q$ consists of exactly two distinct points in $[0, 1] \times \{1\}$.*

Proof. Consider the corresponding ST-left bone $B^{ST}(\sigma_1, \tau_1)$ and its boundary points $(v_1, 1)$ and $(v_2, 1)$. For $i = 1$ and $i = 2$, we will work with the reflections $st_{v_i} \circ st_1$ because they have real distinct critical points $c_1 < c_2 < c_3$. For both $i = 1$ and $i = 2$, the pair of critical d'itineraries at $(1, v_i)$ determines the respective kneading-data $\mathbf{K}_{ST}(1, v_i)$. Note that $\Im_{ST}(\gamma_1)(v_1, 1) \neq \Im_{ST}(\gamma_1)(v_2, 1)$, so $\mathbf{K}_{ST}(1, v_1) \neq \mathbf{K}_{ST}(1, v_2)$. The kneading-data also satisfies for each i the conditions in the extended version of Thurston's theorem: the kneading sequences are finite and tight and $\mathcal{K}_{ST}(1,v_i)(c_1) = \mathcal{K}_{ST}(1,v_i)(c_3) \neq \mathcal{K}_{ST}(1,v_i)(c_2)$. Hence for each i there exists a point $(w_i', v_i') \in P^Q$ such that $q_{v_i'} \circ q_{w_i'}$ has kneading-data $\mathbf{K}_Q(w_i', v_i') = \mathbf{K}_{ST}(1, v_i)$, and subsequently the same critical d'itineraries as (st_{v_i}, st_1). In consequence:

$$\Im_Q(\gamma_1)(v_i', w_i') = \Im_{ST}(\gamma_1)(v_i, w_1)$$

$$\Im_Q(\gamma_2)(v_i', w_i') = \Im_{ST}(\gamma_2)(v_i, 1) = (\Gamma_2, R_1, L_2, L_1, L_2, L_1, ...).$$

So clearly (v_i', w_i') must be in the left bone $B_L^Q = B^Q$ in P^Q corresponding to $B_L^{ST} = B^{ST}$ in P^{ST}. We also get that the d'itinerary of γ_2 under $(q_{v_i'}, q_{w_i'})$ is $(\Gamma_2, R_1, L_2, L_1, L_2, L_1, ...)$. As (v_i', w_i') are on a left quadratic bone, we have $v_i'w_i' > \frac{1}{16}$, so zero is a repeller for the composition $q_{w_i'} \circ q_{v_i'}$. So, the only way for the d'itinerary of a point to stay indefinitely on L_2 and L_1 is for the point to map to zero after a number of iterates. To be consistent with the required d'itinerary, we need to have $(q_{v_i'} \circ q_{w_i'})(\gamma_2) = 0$ and $q_{w_i'}(\gamma_2) > \frac{1}{2}$, so $q_{w_i'}(\gamma_2) = 1$, and hence $w_i' = 1$, for both $i = 1$ and $i = 2$.

In conclusion: for the two points $(v_1, 1), (v_2, 1) \in \delta B^{ST}$ we found two points $(v_1', 1), (v_2', 1) \in \delta B^Q$ with the same corresponding kneading-data. The two points $(v_1', 1)$ and $(v_2', 1)$ we found in δB^Q are the only two boundary points of B^Q. This follows almost immediately from Thurston's uniqueness. □

2.10. **A more complete description of bones in P^{ST} and P^Q.**

We showed that every bone in P^Q is composed of a bone-arc joining two distinct boundary points, and possible other components with no additional boundary points (i.e., loop components). For the time being, let's fix an order-data $(\sigma_1, \tau_1) \in S_n^2$ and focus on the simple arc component of the corresponding bone. In order to keep notation relatively simple, within this section we will designate this arc-bone by $B^Q = B^Q(\sigma_1, \tau_1)$, which is not entirely wrong, as we will eventually rule out the existence of bone-loops.

We plan to prove that, along B^Q, the crossings with other bones occur in the same order of their combinatorics as the crossings along the corresponding bone $B^{ST} = B^{ST}(\sigma_1, \tau_1) \subset P^{ST}$.

We study first the order of occurrence of the primary and secondary intersections along a bone in P^{ST} with order-data (σ_1, τ_1). To fix our ideas, all proofs and results are developed for left bones $B^{ST} = B_L^{ST}$, hence we will omit writing the index L unless it causes ambiguity.

Fix a stunted left bone $B^{ST} = B^{ST}(\sigma_1, \tau_1)$ and slide (v, w) along B^{ST}. Clearly, $\Im_{ST}(\gamma_1)$ only changes at the primary intersection (v_0, w_0). Therefore, $B_*^{ST} = B^{ST} \backslash \{(v_0, w_0)\}$ can be divided into two halves, each corresponding to a different d'itinerary of γ_1 under (st_v, st_w); call B_-^{ST} the left half, containing the boundary point $(v_1, 1) \in \delta B^{ST}$, and B_+^{ST} the one containing $(v_2, 1) \in \delta B^{ST}$ (where $v_1 < v_2$):

$$B^{ST} = B_*^{ST} \cup \{(v_0, w_0)\} = B_-^{ST} \cup \{(v_0, w_0)\} \cup B_+^{ST}.$$

To fix our ideas, we look at B_-^{ST}; the results and their proofs should work symmetrically for B_+^{ST}. B_-^{ST} is composed of a vertical segment and a horizontal one:

$B_-^{ST} = B_{-,v}^{ST} \cup B_{-,h}^{ST}$,
where $B_{-,v}^{ST} = \{v_1\} \times [w_0, 1]$ and $B_{-,h}^{ST} = [v_1, v_0] \times \{w_0\}$.

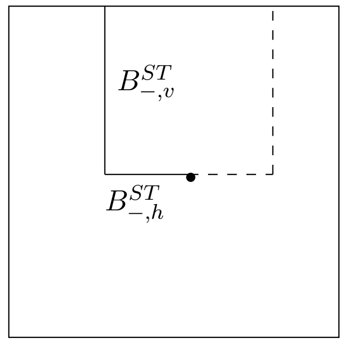


FIGURE 7. *We divide the left half B_{-}^{ST} of a left bone in P^{ST} into a vertical segment $B_{-,v}^{ST}$ and a horizontal segment $B_{-,h}^{ST}$. All secondary intersections occur along $B_{-,v}^{ST}$. All points along the horizontal part are capture points.*

We can now state the following:

Lemma 2.10.1. *The secondary intersections occur along $B_{-,v}^{ST}$ in the strictly decreasing order of their d'itinerary $\Im_{ST}(\gamma_2)$, as w decreases from 1 to w_0.*

Proof. For a fixed $m \geq 1$, call $\mathcal{D}_{ST}^m$ the set of all parameters (v, w) (secondary intersections and capture points) on $B_{-,v}^{ST}$ for which γ_2 maps to either γ_1 in $2m-1$ iterates or to γ_2 in $2m$ iterates. Call $\mathcal{D}_{ST} = \bigcup_{m\geq 1} \mathcal{D}_{ST}^m$ the *distinguished* points on $B_{-,v}^{ST}$. Also call $\Im_{ST}^m(\gamma_2)$ the d'itinerary $\Im_{ST}(\gamma_2)$ truncated to the first $2m$ positions.

As w decreases from 1 to w_0, $\Im_{ST}^m(\gamma_2)$ decreases (in the order inherited from the total order on infinite d'itineraries), with actual changes at all points in $\bigcup_{k\leq m} \mathcal{D}_{ST}^k$. Hence $\Im_{ST}(\gamma_2)$ decreases, with changes at all points in $\mathcal{D}_{ST}$.

Subsequently, $\Im_{ST}(\gamma_2)$ decreases strictly on the set of distinguished points, in particular on the set of secondary intersections (see [17] for details). □

Remark. The theorem makes it possible to identify the order of occurrence of the distinguished points (in particular of the secondary intersections) along $B_{-,v}^{ST}$ by looking at the d'itinerary of γ_2. From the construction of the stunted bones it is also easy to see that there are no secondary intersections on the horizontal segment of $B_{-,h}^{ST}$. In fact, all points of $B_{-,h}^{ST}$ are capture points, and $\Im_{ST}(\gamma_2)(v, w_0)$ is constant for $v \in [v_1, v_0]$.

We shift our focus now to the parameter space P^Q. The corresponding left bone B^Q is a connected arc joining two boundary points $(v_1', 1)$ and $(v_2', 1)$ (with $v_1' < v_2'$) and having a unique primary intersection (v_0', w_0'). As before, the d'itinerary of γ_1 under $(q_{v'}, q_{w'})$ changes only at (v_0', w_0') as we move (v', w') along B^Q. Hence we can divide B^Q into two halves: left of (v_0', w_0'), containing $(v_1', 1)$, and right of (v_0', w_0'), containing $(v_2', 1)$.

$$B^Q = B_{-}^Q \cup \{(v_0', w_0')\} \cup B_{+}^Q$$

We will study the left half, by comparison with the vertical left half $B_{-,v}^{ST}$.

We know that there is a bijective correspondence between secondary intersections along $B_{-,v}^{ST}$ and B_-^Q that associates to each intersection in $B_{-,v}^{ST}$ one with $\Im_Q(\gamma_2) = \Im_{ST}(\gamma_2)$ in B_-^Q. We would like to prove that these secondary intersections occur on both $B_{-,v}^{ST}$ and B_-^Q in the same decreasing order of $\Im(\gamma_2)$, going from the boundary towards the primary intersection. In other words, we prove that the bijection is order preserving.

Fix $m \geq 1$. By analogy, we call $\Im_Q^m(\gamma_2)$ the d'itinerary $\Im_Q(\gamma_2)$ truncated to the first $2m$ positions. Also, $\mathcal{D}_Q^m$ will be the set of all parameters (v', w') on B_-^Q for which γ_2 maps to either γ_1 in $2m-1$ iterates or to γ_2 in $2m$ iterates, and $\mathcal{D}_Q = \bigcup_{m\geq 1} \mathcal{D}_Q^m$ will stand for the set of *distinguished* points on B_-^Q.

Take the first distinguished point $(l_1, m_1) \in \bigcup_{k\leq m} \mathcal{D}_Q^k$ on B_-^Q (from $(v_1', 1)$ along the connected curve, with the regular order inherited by the order on $(0,1) \subset \mathbb{R}$). We know that there exists a corresponding distinguished point $(\alpha, \beta) \in \bigcup_{k\leq m} \mathcal{D}_{ST}^k \subset B_{-,v}^{ST}$ with the same critical d'itineraries:

1. $\Im_{ST}(\gamma_1)(\alpha,\beta) = \Im_Q(\gamma_1)(l_1, m_1)$.
2. $\Im_{ST}(\gamma_2)(\alpha,\beta) = \Im_Q(\gamma_2)(l_1, m_1)$.

Claim. (α, β) is the first point to occur in $\bigcup_{k\leq m} \mathcal{D}_{ST}^k$ along $B_{-,v}^{ST}$.

Suppose not. Then there exists a point $(v^*, w^*) \in \bigcup_{k\leq m} \mathcal{D}_{ST}^k$ between the boundary point $(v_1, 1)$ and (α, β). We then have:

$$\Im_{ST}^m(\gamma_2)(v_1, 1) > \Im_{ST}^m(\gamma_2)(v^*, w^*) > \Im_{ST}^m(\gamma_2)(\alpha, \beta)$$

$$\Im_{ST}^m(\gamma_2)(v_1, 1) = \Im_Q^m(\gamma_2)(v_1', 1)$$

$$\Im_Q^m(\gamma_2)(l_1, m_1) = \Im_{ST}^m(\gamma_2)(\alpha, \beta).$$

The contradiction follows easily. (Note, for instance, that the conditions imply that the pair of critical d'itineraries at $(v_1, 1)$ has to be the same as the pair at a point right before (α, β)).

So the distinguished point in $(\alpha, \beta) \in B_-^{ST}$ with d'itinerary $\Im_{ST}(\gamma_2)(\alpha,\beta) = \Im_Q(\gamma_2)(l_1, m_1)$ is the first to occur in $\bigcup_{k\leq m} \mathcal{D}_{ST}^k$. Continuing the procedure shows that the order of occurrence of all points in $\bigcup_{k\leq m} \mathcal{D}_{ST}^k$ along $B_{-,v}^{ST}$ is the same as the order of points in $\bigcup_{k\leq m} \mathcal{D}_Q^k$ along B_-^Q (i.e., the decreasing order of the d'itinerary $\Im^m(\gamma_2)$). We can state this as follows:

Theorem 2.10.1. *For a fixed $m \geq 1$, going along $B_{-,v}^{ST}$ from (v_0, w_0) to $(v_1, 1)$ and along B_-^Q from (v_0', w_0') to $(v_1', 1)$, the d'itinerary $\Im^m(\gamma_2)$ is monotonically increasing, with actual changes occurring at each distinguished point in $\bigcup_{k\leq m} \mathcal{D}_{ST}^k$ and $\bigcup_{k\leq m} \mathcal{D}_Q^k$, respectively.*

A similar argument proves the corresponding statement for boundary points of bones:

Theorem 2.10.2. *For a fixed $m \geq 1$, going along $[0,1] \times \{1\} \subset \partial P^{ST}$ and $[0,1] \times \{1\} \subset \partial P^Q$, the d'itinerary $\Im(\gamma_2) = (\Gamma_2, R_1, L_2, L_1, L_2, L_1, ...)$ stays constant, but the d'itinerary $\Im^m(\gamma_1)$ increases monotonically, with an actual change at each endpoint of a bone of period $2k \leq 2m$.*

2.11. **The big picture.** **Overview of results**:

Fix $m \geq 1$. Going along B_-^{ST} from (v_0, w_0) to $(v_1, 1)$ and along B_-^Q from (v_0', w_0') to $(v_1', 1)$, the truncated d'itinerary $\Im^m(\gamma_2)$ increases monotonically, with an actual increase at each crossing with a right bone. The same result holds for the right halves B_+^{ST} and B_+^Q. There is a one-to-one correspondence between the crossing points of bones of period at most $2m$ in the two families, correspondence that preserves the order of critical d'itineraries (i.e., of the joint order-data).

Slide from left to right along the upper boundary segment $[0,1] \times \{1\}$ of the two parameter spaces. The d'itinerary $\Im(\gamma_2)$ does not change, and the truncated d'itinerary $\Im^m(\gamma_1)$ increases monotonely, with an actual change at each end-point of a left bone. There is a one-to-one correspondence between all boundary points of bones of period smaller than $2m$ in the two families, correspondence that preserves the order of the critical d'itineraries.

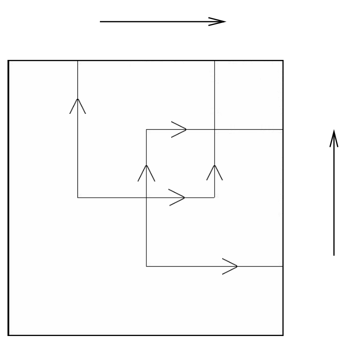
We want to restate the results in terms of kneading-data.

Theorem 2.11.1. *In the parameter space P^Q, the kneading-data of the maps $q_{w'} \circ q_{v'}$ increases along a left bone-arc from its primary intersection towards either boundary point and increases along the upper boundary interval $[0,1] \times \{1\} \in \partial P^Q$ from left to right (see picture). A symmetric statement holds for right bones and the right boundary interval.*

Proof. Consider two arbitrary $(v_1', w_1'), (v_2', w_2') \in B_-^Q$ and the d'itineraries $\Im_Q^i(\gamma_2)$ of γ_2 under $q_{w_i'} \circ q_{v_i'}$, for $i = 1, 2$. If $\Im_Q^1(\gamma_2) < \Im_Q^2(\gamma_2)$, then the kneading data $\mathbf{K}(q_{w_1} \circ q_{v_1}) << \mathbf{K}(q_{w_2} \circ q_{v_2})$. If $(v_1', 1), (v_2', 1) \in [0,1] \times \{1\}$ are such that $\Im_Q^1(\gamma_1) < \Im_Q^2(\gamma_1)$, then $\mathbf{K}(q_1 \circ q_{v_1}) << \mathbf{K}(q_1 \circ q_{v_2})$. $\square$

We know (see for example [16]) that if the kneading-data of two maps f and g are such that $\mathbf{K}(f) << \mathbf{K}(g)$, then the values of their topological entropies are such that $h(f) \leq h(g)$. Hence:

Theorem 2.11.2. *The topological entropy increases in P^Q along each bone-arc from its primary intersection towards the boundary ∂P^Q and along the boundary segments $[0,1] \times \{1\}$ and $\{1\} \times [0,1]$ towards the upper right corner (see picture).*



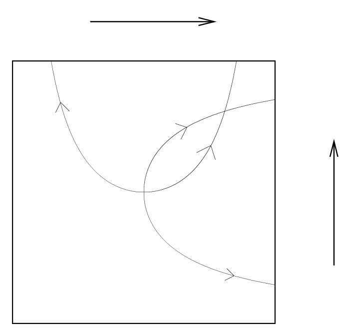


FIGURE 8. *The arrows show the direction of increasing entropy along the bones and the boundary in P^{ST} and P^Q.*

We know now a few things about the combinatorics along the bones, but we have no information about their geometry or degree of smoothness. In Sections 3 and 4, we will obtain two important results addressing these issues. For now, however, we want to point out a few major consequences of our results.

Firstly, let's note that Thurston's uniqueness implies that *for any arbitrary left bone in P^Q, the bone-arc B^Q contains all possible post-critically finite kneading data admissible for the given bone.* In consequence, *any loop component that the bone may have cannot contain any post-critically finite points.* We will use this in our proof in Section 3.

Secondly, let's assume the results that will be proved independently in Sections 3 and 4: the bones in P^Q are smooth $\mathcal{C}^1$ curves, intersecting transversally with each other and with the boundary. There are no bone loops in P^Q, so each bone is a smooth arc connecting two boundary points. Moreover, we have already seen that each such bone-arc contains all post-critically finite kneading-data existing on the corresponding bone in P^{ST}, in the same order of occurrence. Then we obtain a strong topological correspondence between P^{ST} and P^Q as follows.

Definition 2.11.1. Fix $n \in \mathbb{N}$. We define the n-skeleton in either parameter space to be:

S_n^{ST} = the union of all (left and right) bones $B_{2k}^{ST} \subset P^{ST}$ of period $2k \leq 2n$, together with the boundary ∂P^{ST};

S_n^Q = the union of all (left and right) bones $B_{2k}^Q \subset P^Q$ of period $2k \leq 2n$, together with the boundary ∂P^Q.

By a vertex of either skeleton we mean either an end-point of its bones or a (primary or secondary) intersection point.

Theorem 2.11.3. *For any fixed $n \in \mathbb{N}$, there is a homeomorphism:*

$$\eta_n \colon P^{ST} \to P^Q,$$

which maps S_n^{ST} onto S_n^Q, carrying ∂P^{ST} to ∂P^Q and carrying bones to corresponding bones and vertexes to vertexes with the same data.

Proof. The construction of the homeomorphism is topologically natural. Define η_n on the set of vertices by corresponding to each vertex in S_n^{ST} the unique one in S_n^Q with the same data. Along each bone, η_n preserves the order of the vertices. Hence we can extend it continuously to the intervals on the bones or boundary between each two vertices, then to each skeleton-enclosed region. This can easily be done in such a way that the resulting continuous map $\eta_n \colon P^{ST} \to P^Q$ is a homeomorphism. □

We can associate to the n-skeleton in either parameter space a topological cell-structure as follows:

• the 0-cells are points, more precisely the vertexes of the n-skeleton;

• the 1-cells are the connected components of the bones obtained by deleting the vertexes, hence they are homeo to open intervals;

• the 2-cells are the connected components of the complement of the n-skeleton in the respective parameter space, hence they are homeo to open discs.

We will also use the closures of such cells, which are homeo to points, closed intervals and closed discs respectively.

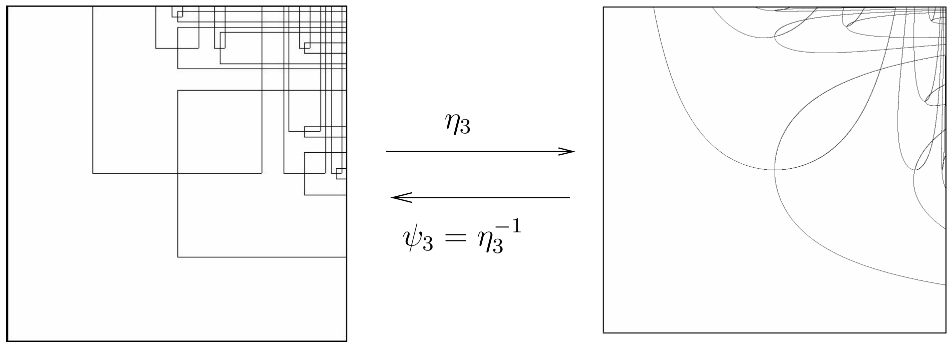


FIGURE 9. *The n-skeletons define topological cell-complexes in both parameter spaces. The map η_n is a homeomorphism between these complexes. The picture illustrates $n = 3$.*

We call the resulting complexes P_n^{ST} in P^{ST} and P_n^Q in P^Q. The map $\eta_n\colon P_n^{ST} \to P_n^Q$ is a *homeomorphism of cell complexes*, taking each cell in P_n^{ST} to a corresponding cell in P_n^Q by carrying vertexes to vertexes with the same entropy and edges to edges with the same interval of entropies.

3. **Hyperbolicity in P^Q.**

3.1. **The mapping schema of a hyperbolic map.**

Definition 3.1.1. Let M be a finite disjoint union of copies of $\mathbb{C}$, and let $f : M \longrightarrow M$ be a proper holomorphic map of degree ≥ 2 on each component of M. We say that f is *hyperbolic* if every critical orbit converges to an attracting cycle.

Let f be a hyperbolic map as above. Let $W(f)$ be the union of the basins of attraction of all attracting cycles of f. f carries each component $W_\alpha \subset W(f)$ onto a component W_β by a map of degree $d_\alpha \geq 1$. Also let $W^c(f)$ be the union of all critical components $W_\alpha \subset W(f)$ (i.e., of all components W_α that contain critical points of f).

We define the *reduced mapping schema* $\overline{S}(f) = (|S|, F, w)$ associated to f as the triplet made of:

• a set of vertices $|S|$, obtained by associating a vertex α to each critical component $W_\alpha \subset W^c(f)$;

• a weight function $w\colon |S| \to |S|$, defined as $w(\alpha) =$ the number of critical points of f in W_α;

• a set of edges $F\colon |S| \to |S|$, $F(\alpha) = \beta$, where W_β is the image of W_α under the first return map to $W^c(f)$.

The *critical weight* of $\overline{S}(f)$ is defined as $w(f) = \sum_\alpha w(\alpha)$.

All hyperbolic maps that interest us have reduced mapping schemata of critical weight 2, so we will only look at the cases that appear for $w = 2$. For a more general analysis, see [11].

To a fixed mapping schema with $w = 2$, we associate the universal polynomial model space $\mathcal{P}$. This will be the space of all maps f from $\mathbb{C}_1 \sqcup \mathbb{C}_2$ to itself such that the restriction of f to each copy of $\mathbb{C}$ is a monic centered polynomial of degree 2. More precisely, for $a_1, a_2 \in \mathbb{C}$:

$f(z) = z^2 - a_1$, for all $z \in \mathbb{C}_1$ and $f(z) = z^2 - a_2$, for all $z \in \mathbb{C}_2$.

We say that a map $f \in \mathcal{P}$ belongs to the *connectedness locus* $\mathcal{C}$ if its filled Julia set $K(f)$ intersects both $\mathbb{C}_1$ and $\mathbb{C}_2$ in a connected set. The *hyperbolic connectedness locus* $\mathcal{H} \subset \mathcal{C}$ is the open set of all $f \in \mathcal{P}$ for which the orbits of both critical points $0 \in \mathbb{C}_1$ and $0 \in \mathbb{C}_2$ converge to attracting periodic orbits.

Such hyperbolic maps can be roughly classified into the three following types (see [13]):

(1) **Bitransitive case**: $0 \in U_1 \subset \mathbb{C}_1$ and $0 \in U_2 \subset \mathbb{C}_2$ such that U_1 is mapped to U_2 under q_1 iterates of f, and U_2 is mapped back to U_1 under q_2 iterates.

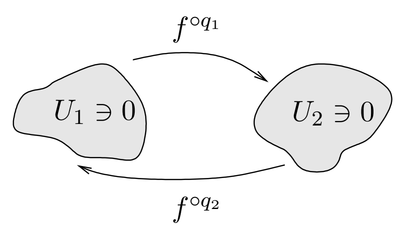


Figure 10. *The behavior of a bitransitive hyperbolic map.*

(2) **Capture case**: $0 \in U_1 \subset \mathbb{C}_1$ and $0 \in U_2 \subset \mathbb{C}_2$ such that U_1 is periodic and U_2 is not, but some forward image of U_2 coincides with U_1. Also its symmetric case.

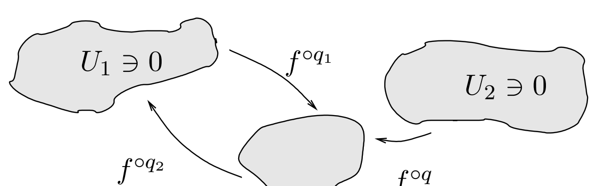


Figure 11. *The behavior of a map in the capture case.*

(3) **Disjoint periodic sinks**: $0 \in U_1$ and $0 \in U_2$, where U_1 and U_2 are periodic of periods q_1 and q_2, but no forward image of U_1 coincides with U_2 and vice versa.

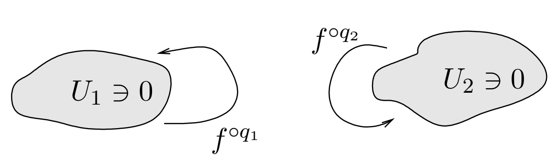


Figure 12. *The behavior of a map in the disjoint sinks case.*

For maps $f \in \mathcal{H}$, we may consider their reduced mapping schemata $\overline{S}(f)$. These schemata will all have critical weight 2, but not all are isomorphic (see Figure

14). However, all maps in each connected component of $\mathcal{H}$ clearly have isomorphic schemata. Furthermore, by Theorem 4.1 in [11]:

Theorem 3.1.1. *If $H_\alpha \subset \mathcal{C}$ is a hyperbolic component of $\mathcal{H}$ with maps having reduced schemata isomorphic to S, then H_α is diffeomorphic to a model space $B(S)$. In particular, any two hyperbolic components H_α and H_β with schemata isomorphic to S are diffeomorphic. Moreover, each H_α contains a unique post-critically finite map f_α, called its center.*

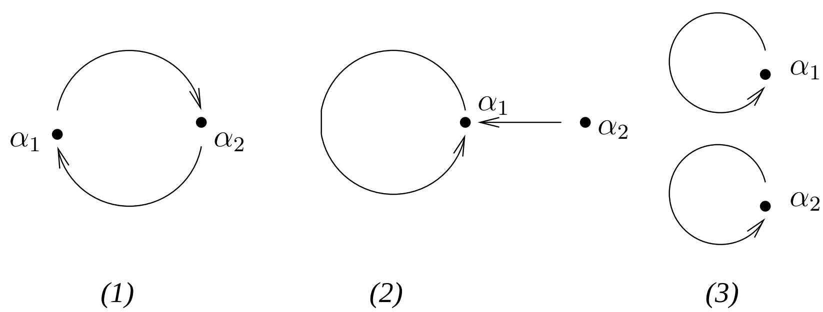


FIGURE 13. *Mapping schemata of weight $w = 2$. (1) Bitransitive case: $|S| = \{\alpha_1, \alpha_2\}$, $F(\alpha_1) = \alpha_2$, $F(\alpha_2) = \alpha_1$, $\omega(\alpha_1) = \omega(\alpha_2) = 1$. (2) Capture case: $|S| = \{\alpha_1, \alpha_2\}$, $F(\alpha_1) = \alpha_1$, $F(\alpha_2) = \alpha_1$, $\omega(\alpha_1) = \omega(\alpha_2) = 1$. (3) Disjoint sinks case: $|S| = \{\alpha_1, \alpha_2\}$, $F(\alpha_1) = \alpha_1$, $F(\alpha_2) = \alpha_2$, $\omega(\alpha_1) = \omega(\alpha_2) = 1$*

Definition 3.1.2. A real form of the mapping schema S is an antiholomorphic involution $\rho\colon \mathbb{C}_1 \sqcup \mathbb{C}_2 \to \mathbb{C}_1 \sqcup \mathbb{C}_2$, which commutes with the special map $f_0^S\colon \mathbb{C}_1 \sqcup \mathbb{C}_2 \to \mathbb{C}_1 \sqcup \mathbb{C}_2$, $f_0^S(z) = z^2$. The collection of maps $f \in \mathcal{P}$ that commute with ρ is an affine space $\mathcal{P}_\mathbb{R}(\rho)$, which we call the real form of $\mathcal{P}$ associated with ρ. We also define the corresponding real connectedness locus and the real hyperbolic locus as:
$\mathcal{C}_\mathbb{R}(\rho) = \mathcal{C} \cap \mathcal{P}_\mathbb{R}(\rho)$
$\mathcal{H}_\mathbb{R}(\rho) = \mathcal{H} \cap \mathcal{P}_\mathbb{R}(\rho)$.

For each mapping schema of weight 2, there are exactly two real forms. The form $\rho_0(z) = \overline{z}$ corresponds to the space $\mathcal{P}_\mathbb{R}(\rho_0)$ of real polynomials in $\mathcal{P}$. If we restate Theorem 6.4 of [11] in our particular case, we obtain:

Theorem 3.1.2. *Any hyperbolic component in $\mathcal{C}_\mathbb{R} = \mathcal{C}_\mathbb{R}(\rho_0) \subset \mathcal{P}_\mathbb{R}(\rho_0)$ is a topological 2-cell with a unique "center point" and is real analytically homeomorphic to a model space $B_\mathbb{R}(S, \rho_0)$.*

In other words, all hyperbolic components with the same schemata in $\mathcal{C}_\mathbb{R}$ are diffeomorphic to each other. For example, all bitransitive components are diffeo to the principal component centered at

$$f_0^S : \mathbb{C}_1 \sqcup \mathbb{C}_2 \longrightarrow \mathbb{C}_1 \sqcup \mathbb{C}_2, \;\; f_0^S(z) = z^2.$$

For a detailed characterization of the construction and properties of the suitable model spaces, see [11].

3.2. **Hyperbolic components in P^Q.**

Let us return to our space, containing real quartic polynomials that are compositions $q_w \circ q_v$ of logistic maps.

Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two copies of the complex plane. Fix $(v,w) \in [0,1]^2$ and consider $q_v \colon \mathbb{C}_1 \to \mathbb{C}_2$ and $q_w \colon \mathbb{C}_2 \to \mathbb{C}_1$ as the complex extensions of two fixed logistic maps of the interval. We define a new map: $q_w^v \colon \mathbb{C}_1 \sqcup \mathbb{C}_2 \to \mathbb{C}_1 \sqcup \mathbb{C}_2$, acting as q_v on $\mathbb{C}_1$ and as q_w on $\mathbb{C}_2$.

Let $W(q_w^v) \subset \mathbb{C}_1 \sqcup \mathbb{C}_2$ be the open set consisting of all complex numbers in $\mathbb{C}_1$ and $\mathbb{C}_2$ whose forward orbit under q_w^v converges to an attracting periodic orbit of q_w^v.

Under iteration of q_w^v, each component of $W(q_w^v)$ is mapped onto a component of $W(q_w^v)$. As before, we will say that q_w^v is hyperbolic if both $\gamma_1 \in I_1 \subset \mathbb{C}_1$ and $\gamma_2 \in I_2 \subset \mathbb{C}_2$ are contained in $W(q_w^v)$.

It would be convenient to find a correspondence between our family of pairs of real quadratic maps, parametrized by $(v,w) \in P^Q$ and the family of degree 2 normal polynomials. It can be shown that each map $q_w \circ q_v \colon \mathbb{C}_1 \to \mathbb{C}_2$ is conjugated by a complex affine map L to a composition of maps $z \to z^2 - a_1$ and $z \to z^2 - a_2$. Moreover, the correspondence $(v,w) \to (a_1,a_2)$ is "nice" enough to permit us to carry over to P^Q properties we have in the space of normal forms. More precisely:

Theorem 3.2.1. *Let $\mathcal{U}$ be the subset of P^Q consisting of pairs (v,w) with $vw > \frac{1}{16}$. For each such pair $(v,w) \in \mathcal{U}$, there is a unique pair $(A,B) \in \mathbb{R}^2$ such that $q_w \circ q_v$ is linearly conjugate to $z \to z^4 + Az^2 + B$; there also exists a unique pair $(a_1,a_2) \in \mathbb{R}^2$ so that $q_w \circ q_v$ is linearly conjugate to the composition of $z \to z^2 - a_1$ and $z \to z^2 - a_2$.*

Furthermore, recall that the connectedness locus $\mathcal{C}_\mathbb{R} \subset \mathbb{R}^2$ is the subset of parameters $(a_1,a_2) \in \mathbb{R}^2$ for which the complex critical points of $(z^2 - a_1)^2 - a_2$ have bounded orbits. The correspondence described above,

$$\Xi \colon \mathcal{U} \to \mathcal{C}_\mathbb{R}$$

$$\Xi(v,w) = (a_1,a_2),$$

is a bijective diffeomorphism.

Proof. Each $q_w \circ q_v$ with $(v,w) \in P^Q$ is conjugated by an affine map $L(z) = -\frac{8}{\sqrt[3]{v^2 w}} z + \frac{1}{2}$ to a composition of the two monic centered quadratic complex maps: $z \to \zeta = z^2 - a_1(v,w)$ and $\zeta \to z = \zeta^2 - a_2(v,w)$. The correspondence

$$\Xi : \mathcal{U} \to \mathbb{R}^2,\ \Xi(v,w) = (a_1(v,w), a_2(v,w))$$

is a diffeomorphism onto its image, where the image $\Xi(\mathcal{U})$ is exactly the real connectedness locus $\mathcal{C}_\mathbb{R}$ in $\mathcal{P}_\mathbb{R}$. $\square$

Remarks. (1) The region $P^Q \setminus \overline{\mathcal{U}} = \{(v,w) \text{ with } vw < \frac{1}{16}\}$ is itself a hyperbolic component of P^Q, whose maps have all critical points attracted to zero. The map Ξ folds this region and the principal component centered at $(v,w) = (\frac{1}{2}, \frac{1}{2}) \in P^Q$ onto the same component in $\mathcal{C}_\mathbb{R}$.

(2) *All bones in P^Q are contained in $\mathcal{U}$.* Indeed, suppose $(v,w) \notin \mathcal{U}$. The fixed origin is not repelling for the map $q_w \circ q_v$ with negative Schwarzian derivative, so it attracts all critical points, and hence (v,w) can't be on a bone.

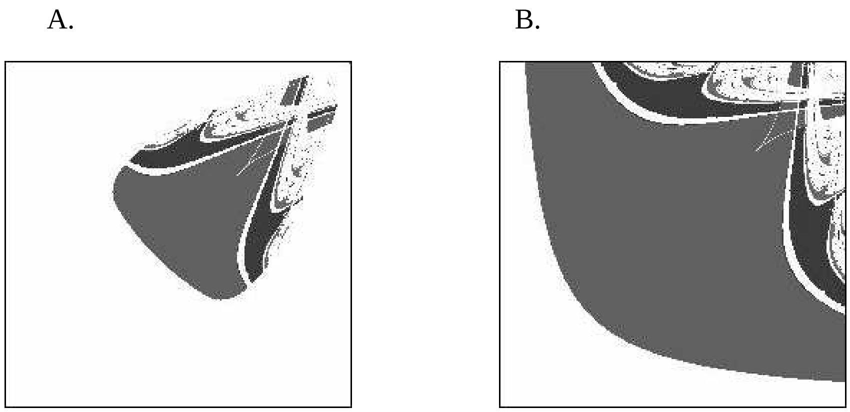


FIGURE 14. *A. Hyperbolic components in $\mathbb{R}^2$ for the classical family of pairs of quadratic monic centered maps. The picture shows the parameter window $(a_1, a_2) \in [-2, 2] \times [-2, 2]$. B. Hyperbolic components in $\mathcal{U} \subset P^Q$. The principal component in both cases is visible as the large central dark region.*

We use the terminology in Section 2 to give the needed description of the hyperbolic components in our original parameter space P^Q. Hyperbolic components within each class (bitransitive, capture and disjoint sinks) are diffeomorphic to each other. The center points in each case will be respectively a primary intersection, a capture point, or a secondary intersection.

Theorem 3.2.2. *Each hyperbolic component in $\mathcal{U} \subset P^Q$ is a topological 2-cell that contains a unique post-critically finite point, called its center. Moreover, every bone that intersects such a component does it along a simple arc passing through the center. Subsequently, there could be either one bone crossing the component through its center (capture case) or a pair of left-right bones intersecting transversally at the center point (bitransitive and disjoint sinks cases).*

For more details on the proof of Theorem 3.2.2, see [17].

3.3. Density of hyperbolicity in P^Q.

Theorem 3.3.1. *Hyperbolicity is dense in the parameter space P^Q.*

Remark. The proof of this density result in the context of the present paper is an adaptation of the proof of the Fatou conjecture presented in [9]. We will only sketch the main steps; for complete arguments, see the cited references. One of the main results in [9] is the following *Rigidity Theorem*, which we will also use (in our more restricted context).

Rigidity Theorem. *Let f and f' be two polynomials with real coefficients, real non-degenerate critical points, connected Julia set and no neutral periodic points. If f and f' are topologically conjugate as dynamical systems on the real line $\mathbb{R}$, then they are quasiconformally conjugate as dynamical systems on the complex plane $\mathbb{C}$.*

Proof. We define the family $\mathcal{S}_4$ as the set of complex polynomials $Q\colon \mathbb{C} \to \mathbb{C}$ of degree 4, boundary anchored (i.e., $Q(0) = Q(1) = 0$) and such that $Q(z) = Q(1-z)$, for all $z \in \mathbb{C}$.

Consider X_s to be the subset of maps in $\mathcal{S}_4$ with the following properties:

- They have real coefficients.
- Their three critical points are real and nondegenerate.
- All critical points and values are in [0,1]. Hence, their Julia sets are connected (see for example [14]).
- The boundary $\{0, 1\}$ is repelling.

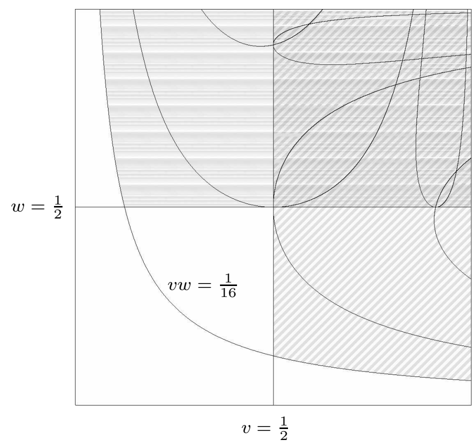


FIGURE 15. *All maps in* $\{vw < \frac{1}{2}\}$ *and in* $\{v < \frac{1}{2},\, w < \frac{1}{2}\}$ *are hyperbolic. Hyperbolic maps are dense in* $\{vw > \frac{1}{16},\, v \geq \frac{1}{2}\}$ *(slant shaded). By symmetry, they are dense in* $\{vw > \frac{1}{16},\, w \geq \frac{1}{2}\}$ *(horizontaly shaded). The region* $vw > \frac{1}{16}$ *contains all left and right bones.*

The three complex critical points of an arbitrary $P \in P^Q$ are C_1, $C_2 = \frac{1}{2}$ and $C_3 = 1 - C_1$. An equivalent condition to $C_1 \in \mathbb{R}$ is that

$$q_v\left(\frac{1}{2}\right) \geq \frac{1}{2} \;\Leftrightarrow\; v \geq \frac{1}{2}.$$

So,

$$X_s = \{q_w \circ q_v, \text{ where } (v, w) \in P^Q,\, v \geq \frac{1}{2},\, vw > \frac{1}{16}\}.$$

We **claim** that hyperbolic polynomials are dense in X_s. Then the proof of 3.3.1 follows relatively easily. Indeed, the claim implies directly density of hyperbolicity in the region in P^Q where $vw > \frac{1}{16}$ and $v \geq \frac{1}{2}$. By the symmetry property (2), the result follows in the region where $vw > \frac{1}{16}$ and $w \geq \frac{1}{2}$. In the regions $\{vw > \frac{1}{16},\, v < \frac{1}{2},\, w < \frac{1}{2}\}$ and $\{vw < \frac{1}{16}\}$, the proof is trivial: if $vw < \frac{1}{16}$, then all three critical diorbits of $q_w \circ q_v$ converge to zero, while if $v < \frac{1}{2}$, $w < \frac{1}{2}$ and $vw > \frac{1}{16}$, then all critical diorbits converge to a point in $(0, \frac{1}{2})$. $\square$

Next, we aim to prove density of hyperbolicity in X_s.

Lemma 3.3.1. *Consider $P \in X_s$ with one parabolic cycle $\{z_1, ..., z_m\}$. We can approximate P by a polynomial $S \in X_s$ for which the cycle is attracting.*

Sketch of proof: Fix $P \in X_s$ as above.

It is fairly easy to show the existence of a polynomial $Q\colon \mathbb{C} \to \mathbb{C}$ with real coefficients and the following properties (see [17]):

- $Q(z) = Q(1-z),\ \forall\, z \in \mathbb{C}$
- $Q(z_j) = 0,\ \forall\, j = \overline{1, m}$
- $Q(0) = Q(1) = 0$
- $Q'(x) = 0$ when $P'(x) = 0$
- $\sum \dfrac{Q'(z_j)}{P'(z_j)} < 0.$

Consider the new polynomial $R = P + \epsilon Q$. For small real values of ϵ, R perturbes the neutral cycle of P to an attracting cycle:

$$\sum \log|R'(z_j)| = \sum \log|P'(z_j)| + \sum \log\left|1 + \epsilon\frac{Q'(z_j)}{P'(z_j)}\right| =$$
$$= \epsilon \sum \frac{Q'(z_j)}{P'(z_j)} + \wr(\epsilon^2) < 0.$$

For small enough values of ϵ, R has the following properties:

- The parabolic cycle of P is attracting for R.
- The attracting/repelling cycles of P change to attracting/repelling cycles for R (hence $\{0\}$ remains a repelling fixed boundary point for R).
- $R(z) = R(1-z),\ \forall\, z \in \mathbb{C}$ and $R(0) = R(1) = 0$ (i.e., $R \in \mathcal{S}^4$).
- R has real coefficients.
- The critical points of R are the same as the critical points of P, hence they are real and nondegenerate; all critical points and values are contained in $[0, 1]$, hence the Julia set $J(R)$ is connected.

However, in order to satisfy all required conditions, the polynomial Q we have found (hence R) may have degree larger than 4. We use the Straightening Theorem to obtain a degree 4 polynomial $S \in X_s$ with the same behavior as R (see for example [4] or [17]). □

For every $Q \in \mathcal{S}_4$, let $\tau(Q)$ be the number of critical points contained in the attracting basin of a hyperbolic attracting cycle of Q. Define

$$X_s' = \{Q \in X_s \ /\ \tau(Q) \text{ has a local maximum at } Q\}.$$

As τ is uniformly bounded above, X_s' is dense in X_s. Moreover, τ is locally constant at any $P \in X_s'$, hence we have the following:

Proposition 3.3.1. *X_s' is open and dense in X_s.*

Proposition 3.3.2. *No map in X_s' has a neutral cycle.*

Proof. Consider $P \in X_s'$ and S given by the lemma. By making the perturbation small enough, we can arrange that the other hyperbolic attractors of P do not disappear. Moreover, we can also make sure that the critical points that were attracted to the attracting cycles remain so under the perturbation.

On the other hand, each attracting cycle attracts at least one critical point. Hence, introducing a new attractor by perturbing P to S will change τ:

$$\tau(S) \geq \tau(P) + 1,$$

a contradiction with the local maximality of τ at P. □

We finish by giving a reduced statement, from which Theorem 3.3.1 should follow now immediately.

Theorem 3.3.2. *Hyperbolic polynomials are dense in X'_s.*

The proof of the reduced statement is detailed in Section 3.4.

3.4. **A reduced density result.**

Recall that two points z_1 and z_2 are in the same foliated equivalence class of a map f if their grand orbits under f have the same closure. For a fixed f, we denote by n_{ac} the number of foliated equivalence classes of acyclic critical points in the Fatou set of f. By [15], the complex dimension of the Teichmuller space of a map $f\colon \mathbb{C} \to \mathbb{C}$ is given by

$$\dim(Teich(f)) = n_{ac} + n_{hr} + n_{lf} + n_p,$$

where:

n_{ac} = the number of foliated equivalence classes of acyclic critical points in the Fatou set $F(f)$;

n_{hr} = the number of Herman rings of f;

n_{lf} = the number of invariant line fields;

n_p = the number of parabolic cycles.

If $P \in X'_s$, P has no Herman rings and no Siegel discs. By [9] and [18], P does not support an invariant line field in its Julia set. We also proved in Lemma 3.3.2 that P does not have any parabolic basins. So, all connected components of its Fatou set are attracting basins.

$$n_{hr} = n_{lf} = n_p = 0 \;\Rightarrow\; \dim(Teich(P)) = n_{ac}.$$

Hence, the set

$$QC(P) = \{Q \in \mathcal{S}_4 \;/\; Q \text{ quasiconformally conjugate to} P\}$$

is covered by countably many complex submanifolds of dimension n_{ac}. Subsequently, the set

$$QC_{\mathbb{R}}(P) = QC(P) \cap X_s$$

is covered by countably many embedded real analytic submanifolds of X_s with real dimension n_{ac}.

We will also use the following([7]):

Definition 3.4.1. Consider two 3-modal maps $P, Q\colon [0,1] \to [0,1]$ with critical points $c_i(P)$ and $c_i(Q)$, for $i \in \{1,2,3\}$. Consider also:
$h_P^Q\colon \bigcup_{n,i} P^n(c_i(P)) \to \bigcup_{n,i} Q^n(c_i(Q))$,
defined by:
$h_P^Q(P^n(c_i(P))) = Q^n(c_i(Q))$, $\forall i \in \{1,2,3\}$ and $\forall n \in \mathbb{N}$.
If h_P^Q is an order-preserving bijection, then we say that P and Q are *combinatorially equivalent* as 3-modal maps of the interval.

The relationship between combinatorial equivalence and topological conjugacy in our space X_s can be described by the following theorem ([7]):

Theorem 3.4.1. *Call $\mathcal{F}$ the family of maps f of the interval satisfying the following:*
(1) they are of class $\mathcal{C}^3$;
(2) they have nonflat critical points (i.e., $D^2 f(c) \neq 0$, $\forall c$ such that $Df(c) = 0$);
(3) they have negative Schwartzian derivative: $Sf < 0$;
(4) the boundary of the interval is repelling (i.e., $|Df(x)| > 1$, if $x \in \{0,1\}$);
(5) they have no one-sided periodic attractors.

Two maps $f, g \in \mathcal{F}$ are topologically conjugate ($f \overset{top}{\sim}_{\mathbb{R}} g$) if and only if they are combinatorially equivalent ($f \overset{c.e.}{\sim}_{\mathbb{R}} g$).

Remark. If P and Q are maps in X_s' restricted to the interval $[0,1]$, then both the conditions of Theorem 3.4.2 and the Rigidity Theorem are satisfied, and hence we have the following equivalences:

$$P \overset{c.e.}{\sim}_{\mathbb{R}} Q \;\Leftrightarrow\; P \overset{top}{\sim}_{\mathbb{R}} Q \;\Rightarrow P \overset{qc.}{\sim}_{\mathbb{C}} Q.$$

Proof of theorem 3.3.1. If we think of $\mathcal{S}_4$ as a subset of $\mathbb{C}^2$, then we can consider the three holomorphic functions $c_i\colon \mathcal{U} \to \mathbb{C}$, for $i \in \{1,2,3\}$, that give the three critical points of each map $Q \in \mathcal{U}$.

Fix $P \in X_s'$. By taking $\mathcal{B} \subset \mathcal{U} \subset \mathcal{S}^4$ to be a small ball around P, we can arrange to have $c_1(Q) < c_2(Q) < c_3(Q) = -c_1(Q)$, for any $Q \in \mathcal{B} \cap X_s$. Take $\mathcal{B}$ small enough for τ to be constant: $\tau = \tau(Q)$, $\forall Q \in \mathcal{B} \cap X_s$ (recall τ is locally constant at each $P \in X_s'$).

We want to prove (by contradiction) that $\mathcal{B} \cap X_s$ contains hyperbolic maps. Suppose the maps in $\mathcal{U} \cap X_s$ are not hyperbolic, hence $\tau < 3$. There are two cases that remain for analysis:

(1) $\tau = 1$ (only C_2 is attracted) or $\tau = 2$ (only C_1 and C_3 are attracted). Either way, there is only one foliated equivalent class of critical points in the Fatou set, hence $n_{ac} \leq 1$ (note that the critical points are not necessarily acyclic). Hence $QC_{\mathbb{R}}(Q)$ is in this case at most a countable union of lines in X_s, for any $Q \in \mathcal{B} \cap X_s$.

(2) $\tau = 0$ (no critical points are attracted). Hence $n_{ac} = 0$, so $QC_{\mathbb{R}}(Q)$ is a countable union of points in X_s, for any $Q \in \mathcal{B} \cap X_s$.

A. Suppose first there are no bones crossing the neighbourhood $\mathcal{B}$.

Let's first remark that if there are no other *critical relations* in $\mathcal{B}$ (i.e., there are no $m, n \in \mathbb{N}$ such that $Q^m(c_1(Q)) = Q^n(c_2(Q))$ for some $Q \in \mathcal{B}$), then for any arbitrary $Q \in \mathcal{B}$ the map h_P^Q defined in 3.4.1 is order preserving. (Note that we do not consider $Q(c_1(Q)) = Q(c_3(Q))$ a critical relation.) Indeed, suppose that h reverses the order of two elements:

$$P^k(c_i(P)) < P^l(c_j(P)) \text{ and } \quad Q^k(c_i(Q)) > Q^l(c_j(Q)).$$

By continuity, there exists a $T \in \mathcal{B}$ such that $T^k(c_i(T)) = T^l(c_j(T))$, a contradiction.

When h_P^Q is order-preserving for any $Q \in \mathcal{B} \cap X_s$, it follows that P is combinatorially equivalent to any $Q \in \mathcal{B} \cap X_s$, and hence P is quasiconformally conjugate to any $Q \in \mathcal{B} \cap X_s$, by the Rigidity Theorem. This contradicts the fact that $QC_{\mathbb{R}}(P)$ is at most a union of countably many lines in X_s.

Clearly, the "no critical relations" condition applies in the case $\tau = 1$ or $\tau = 2$. If $\tau = 0$, it could happen that all neibourhoods of τ, arbitrarily small, contain critical relations. In other words, there could exist a map R arbitrarily close to P that has a critical relation, say $R^m(c_1(R)) = R^n(c_2(R))$.

In this case, consider $\Sigma = \{Q \in \mathcal{B} \cap X_s \,/\, Q^m(c_1(Q)) = Q^n(c_2(Q))\}$. This is a 1-dim curve in $\mathcal{B} \cap X_s$. Since there are no other critical relations on Σ, the map h_R^Q is order-preserving for any $Q \in \Sigma$. Subsequently, all maps in Σ are combinatorially equivalent to R and hence quasiconformally conjugate to R. This contradicts the fact that $QC_{\mathbb{R}}(R)$ is a collection of countably many points in X_s, as $\tau = 0$.

B. $\mathcal{B} \cap X_s$ is crossed by a bone B.

Let $R \in B \cap \mathcal{B} \cap X_s$. Bones can't accumulate at R, or R would be hyperbolic. So, there exists a neighbourhood $\mathcal{V}$ of R, $\mathcal{V} \subset \mathcal{B} \cap X_s$ that intersects no other bones than B. Take $S \in \mathcal{V} \backslash B$, and take $\mathcal{W}$ a neighbourhood of S in $\mathcal{V} \backslash B$. Then the argument at **A.** applies for $\mathcal{W}$ and leads us to a contradiction.

This concludes the proof of Theorem 3.3.1. □

4. Geometric properties of the Q-bones.

4.1. Smoothness of the Q-bones.

As we have stated before, a bone in P^Q is an algebraic variety with two boundary points. In this section, we will prove that:

Theorem 4.1.1. *The bones are smooth $\mathcal{C}^1$ curves that intersect transversally.*

WLOG, fix an arbitrary point (v_0, w_0) on a left bone B_L^Q of order-data $(\sigma, \tau) \in S_n^2$. We want to show that B_L^Q is smooth at (v_0, w_0).

For $h = q_{w_0} \circ q_{v_0}$ extended as a map on the complex plane, γ_1 has a superattracting periodic diorbit of period $2n$. Let $U_h = U_h(\gamma_1)$ be the immediate attracting basin of γ_1. Hence, if $K(h)$ is the filled Julia set of h, then $U_h \subset K(h)$ is a simply connected bounded open neighbourhood of γ_1 that is carried to itself by $h^{\circ n}$. We point out the two cases that could appear, depending on the behavior of the other two (complex) critical points of h, called C_1 and C_3.

Case 1. The map h is hyperbolic (i.e., C_1 and C_3 are attracted).

Proof. Within each hyperbolic component in P^Q, the locus of the maps with a specific superattracting diorbit is a smooth complex manifold. Each intersection of two bones is a center point for some hyperbolic component, and the general theory tells us that these intersections are transverse (see [11]). $\square$

Case 2. The map h is not hyperbolic (i.e., C_1 and C_3 are not attracted to attracting cycles).

Proof. We will use quasiconformal surgery in the neighbourhood of our fixed map $h \in P^Q$. No iterates of the other two critical points of h belong to U_h, the immediate attracting basin of γ_1, hence U_h is isomorphic to the open unit disc, parametrized by its Bottcher coordinate. That is, there exists a biholomorphic isomorphism that conjugates $h^{\circ n}$ to the squaring map $z \to z^2$:

$$\beta\colon U_h \to \mathbb{D}, \quad \beta(h^{\circ n}(z)) = (\beta(z))^2.$$

We want to replace the superattracting basin U_h by a basin with small positive multiplier Λ. For each Λ in a small disc centered at zero, we will construct a new map $h_\Lambda = q_{w_\Lambda} \circ q_{v_\Lambda}$, $(v_\Lambda, w_\Lambda) \in P^Q$ in such a way that $\Lambda \to (v_\Lambda, w_\Lambda)$ is analytic and $h_0 = h$.

The composition of smooth (analytic) maps

$$\Lambda \to (v_\Lambda, w_\Lambda) \in P^Q \to m(h_\Lambda)$$

is the identity. (Here m denotes again the function that assigns to each map in P^Q its multiplier at the specified attracting point.) It follows that the partial derivatives $\dfrac{\partial m}{\partial v}$ and $\dfrac{\partial m}{\partial w}$ can't be simultaneously zero on a small neighbourhood of $(v_0, w_0) \in P^Q$. By the Implicit Function Theorem, the bone curve is smooth $\mathcal{C}^1$ on a small neighbourhood of (v_0, w_0). $\square$

4.2. **Quasiconformal surgery construction.**

Consider the map $f(z) = z^2$ on the open unit disk $\mathbb{D}$ (which is the Bottcher parametrization of $h^{\circ n}$). Its unique critical point is the origin. Fix a small $\epsilon > 0$ (along the proof we will make specific requirements of how small we want ϵ to be), and let Λ be an arbitrary complex number such that $0 \leq |\Lambda| \leq \epsilon$.

Using a partition of unity, we perturb the map f to a new degree 2 map g_Λ such that:

• g_Λ has the same dynamics as $f_\Lambda(z) = z^2 + \Lambda z$ inside a small disc around zero; in particular, the origin will be fixed, with multiplier Λ;

• g_Λ has the same dynamics as $f(z) = z^2$ outside a larger disc around zero.

We do this by choosing a radius r with $\dfrac{\epsilon}{2} \leq r \leq min(\dfrac{1}{2}, 1-\epsilon)$, so that f_Λ maps Δ_{r^2} into itself and the critical point of f_Λ is in Δ_{r^2}. We then construct a $\mathcal{C}^1$ partition of unity $\rho\colon \mathbb{C} \to \mathbb{R}$ with $\rho = 0$ outside $\Delta_{\frac{r}{2}}$, $\rho = 1$ inside Δ_{r^2}, and $0 \leq \rho \leq 1$ on $\Delta_{r/2} \backslash \Delta_{r^2}$.

We define $g_\Lambda\colon \mathbb{C} \to \mathbb{C}$ as:

$$g_\Lambda(z) = z^2 + \Lambda \rho(z) z.$$

By asking that $\frac{r}{2}(\frac{r}{2}+\epsilon) \le r^2$ (i.e., $\frac{2\epsilon}{3} \le r$), and by making ϵ smaller if necessary, we make sure that g_Λ has no critical point outside Δ_{r^2}, for any $0 \le |\Lambda| \le \epsilon$. (Recall that the critical point of $g_0(z) = f(z) = z^2$ is $0 \in \Delta_{r^2}$ and the dependence $\Lambda \to g_\Lambda$ is smooth for $|\Lambda| \le \epsilon$.)

In a nutshell: For any fixed $|\Lambda| \le \epsilon$, the map $g_\Lambda \colon \mathbb{D} \to \mathbb{D}$ obtained by this construction is a 2-to-1 $\mathcal{C}^1$ smooth map that carries $\Delta_r \backslash \Delta_{r^2}$ into Δ_{r^2} and Δ_{r^2} into itself. It coincides with f_Λ inside Δ_{r^2} and with f outside of $\Delta_{\frac{r}{2}}$ (in particular it is conformal outside Δ_r) and has no critical points in $\Delta_r \backslash \Delta_{r^2}$. We would like to emphasize that, as $\Delta_r \backslash \Delta_{r^2}$ is mapped by g_Λ directly into Δ_{r^2}, the annulus $\Delta_r \backslash \Delta_{r^2}$ is intersected at most once by any orbit under g_Λ.

We pull g_Λ back to U_h through the Bottcher biholomorpic diffeomorphism β:

$$G_\Lambda = \beta^{-1} \circ g_\Lambda \circ \beta \, : U_h \, \to \, U_h.$$

The new map G_Λ is 2-to-1 and $\mathcal{C}^1$ smooth, and it has similar properties as the ones stated above for g_Λ (see figure):

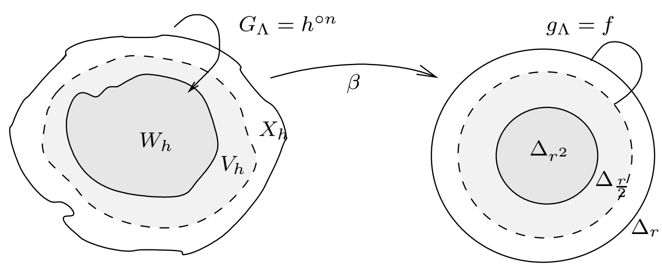


FIGURE 16. *X_h, V_h and W_h are the preimages under the Bottcher map β of Δ_r, $\Delta_{\frac{r}{2}}$ and Δ_{r^2}, respectively. The map $g_\Lambda : \mathbb{D} \to \mathbb{D}$ pulls back as the $\mathcal{C}^1$-map G_Λ, which acts as $h^{\circ n}$ outside V_h and carries V_h to W_h.*

But $h \colon \mathbb{C} \to \mathbb{C}$ carries $U_h \to h(U_h) \xrightarrow{\sim} \ldots \xrightarrow{\sim} h^{\circ(n-1)}(U_h) \xrightarrow{\sim} h^{\circ n}(U_h) = U_h$ (acting as a diffeo except on U_h).

So, we can define H_Λ as:

$H_\Lambda = h$ outside V_h and $H_\Lambda = h^{\circ(1-n)} \circ G_\Lambda$ inside X_h.

The new H_Λ is $\mathcal{C}^1$ (notice that its two expressions coincide on $X_h \backslash V_h$) and has the desired dynamical behavior. However, it may fail to be analytic, hence it may not be a map in P^Q. The rest of the construction aims to transform H_Λ into a polynomial $h_\Lambda \in P^Q$, preserving the dynamics.

The Beltrami dilatation of H_Λ is:

$$\mu_{H_\Lambda}(z) = \frac{(H_\Lambda)_{\overline{z}}}{(H_\Lambda)_z}.$$

Recall that g_Λ has no critical point in $\Delta_r \backslash \Delta_{r^2}$, so $(g_\Lambda)_z \neq 0$ on $\Delta_r \backslash \Delta_{r^2}$. Hence, the denominator of

$$\mu_{g_\Lambda}(z) = \frac{(g_\Lambda)_{\overline{z}}}{(g_\Lambda)_z}$$

never vanishes. Moreover, for fixed z, both top and bottom above are linear in Λ, so it follows easily that

$$\Lambda \to \mu_{g_\Lambda}$$

is an analytic dependence. Hence, $\mu_{H_\Lambda}(z)$ depends itself analytically on Λ, and:

$$\mu_{H_\Lambda}(z) = \mu_{G_\Lambda}(z) = \mu_{g_\Lambda}(\beta(z))\frac{\overline{\beta'(z)}}{\beta(z)} \text{ on } X_h$$
$$\mu_{H_\Lambda}(z) = 0 \text{ outside } V_h.$$

Under iteration of g_Λ, points hit the annulus $\Delta_r \backslash \Delta_{r^2}$ at most once, hence μ_{g_Λ} is bounded less than 1 in modulus. Automatically:

$$|\mu_{H_\Lambda}(z)| = |\mu_{g_\Lambda}(\beta(z))| \left|\frac{\overline{\beta'(z)}}{\beta'(z)}\right| = |\mu_{g_\Lambda}(\beta(z))| \le 1 \text{ on } X_h \backslash W_h \text{ and}$$
$$\mu_{H_\Lambda}(z) = 0 \text{ outside } X_h \backslash W_h.$$

We define an ellipse field starting with circles inside W_h and outside all preimages of X_h under H_Λ and pulling it back invariantly under H_Λ. All orbits hit $X_h \backslash W_h$ (the annular region where H_Λ is not analytic) at most once, so the ellipse field is distorted at most once along any orbit. Let μ_Λ be the coefficient of this field. The dependence of μ_Λ on Λ is holomorphic on $|\Lambda| \le \epsilon$.

Let ϕ_Λ solve the Beltrami equation:

$$\frac{\phi_{\overline{z}}}{\phi_z} = \mu_\Lambda,$$

determined uniquely by the normalization $\phi_\Lambda(0) = 0$, $\phi_\Lambda(1) = 1$, $\phi_\Lambda(\infty) = \infty$.

With this choice for ϕ_Λ, $h_\Lambda = \phi_\Lambda \circ H_\Lambda \circ \phi_\Lambda^{-1}$ is a quartic complex polynomial. Moreover, for $\Lambda \in \mathbb{R}$, $|\Lambda| < \epsilon$, h_Λ corresponds to a pair in the Q -family (see [17]).

4.3. **The impossibility of bone-loops.**

Our plan for this section is to prove that bones in the parameter space P^Q can't contain any loops (i.e., simple closed curves). Recall that we proved in Section 2 that each bone contains a simple bone-arc connecting two boundary points, and that all possible distinguished kneading data of the bone can be found in a certain order along this bone-arc.

We argue by contradiction. Suppose there exists a bone loop L. We will show next that the interior $\mathcal{U}$ of the loop can't contain any hyperbolic maps. This will contradict the genericity of hyperbolicity stated in Theorem 3.3.1.

Remark. The following statements and proofs are given for left bones, but they apply by symmetry to right bones.

Lemma 4.3.1. *A left bone loop in P^Q can't contain any distinguished point, hence it can't contain any crossing with a right bone.*

Proof. Any distinguished point on the loop L would need to have a kneading-data already achieved along the bone arc. Thurston's Theorem shows easily that this is impossible. ∎

Theorem 4.3.1. *The region enclosed by a left bone loop in P^Q can't contain any hyperbolic maps.*

Proof. We know by Theorem 3.2.2 that each hyperbolic component in P^Q is an open topological 2-cell that contains a unique post-critically finite point, called *center*. Moreover, the intersection of any bone with a hyperbolic component must be a simple arc passing through the center.

Suppose, by contradiction, that some hyperbolic component $\mathcal{H}$ intersects the region $\mathcal{U}$. We have two cases:

(1) $\mathcal{H} \subset \mathcal{U}$. Then there is a bone that passes through the center of $\mathcal{H}$. This can only be a bone arc, as bone loops can't contain distinguished points (by Lemma 4.3.1). From the Jordan Curve Theorem, this bone arc has to intersect the bone loop L, a contradiction with Lemma 4.3.1.

(2) $\mathcal{H}$ intersects the loop L. Then the loop must contain the center point of $\mathcal{H}$, again a contradiction. □

5. **Topological conclusions.**

5.1. **The entropy and the bones.**

In Section 5 we will obtain the main result of this paper: for each fixed $h_0 \in [0, \log 4]$, the level-set $\{h = h_0\}$ of the entropy function in either parameter space is connected.

In the ST-family, the analysis of the properties of entropy level-sets is an easy exercise. The following is straightforward (see [16] and [17]):

Theorem 5.1.1. *In P^{ST}, the entropy is a monotone function of either coordinate. For each $h_0 \in [0, \log 4]$, the corresponding h_0-isentrope is contractible, as it is a deformation retract of the contractible region $\{h \leq h_0\}$.*

We want to obtain similar results for quartic polynomials $q_w \circ q_v$, with $(v, w) \in P^Q$. We will need some notations and results from the general theory of m-modal maps of the interval.

If $f :\rightarrow I$ is an m-modal map with folding points $c_1 \leq c_2 \leq ... \leq c_m$, then we define the sign of the fixed point x of $f^{\circ k}$ with itinerary $\Im(x) = (A_0, A_1, ..., A_{k-1})$ as the number

$$\text{sign}(x) = \epsilon(A_0)\epsilon(A_1)...\epsilon(A_{k-1}),$$

where $\epsilon(A_j) = +1,\ -1$ or 0 according to A_j being an increasing/decreasing lap of f or a folding point $c_1, ..., c_m$. If $\text{sign}(x) = -1$, then we say that x is a fixed point of negative type of $f^{\circ k}$.

We define $\text{Neg}(f^{\circ k})$ as the number of fixed points of negative type of $f^{\circ k}$.

Theorem 5.1.2. *If f is an interval m-modal map, then its topological entropy is:*

$$h(f) = \overline{\lim_{k \to \infty}} \frac{1}{k} \log^+(\text{Neg}(f^{\circ k})),$$

where $\log^+ s = \max(\log(s), 0)$.

Proof. See, for example, [16]. □

Remark: $\mathrm{Neg}(f^{\circ k})$ is an integer ≥ 1 unless $f^{\circ k}$ has no fixed points of negative type; in that case, $\log^+(\mathrm{Neg}(f^{\circ k})) = 0$.

The following result is a simple consequence of Theorem 5.1.2 (see [17] for proof and details).

Lemma 5.1.1. *If for two m-modal interval maps f and g the topological entropies $h(f) \neq h(g)$, then the sequence $|\mathrm{Neg}(f^{\circ k}) - \mathrm{Neg}(g^{\circ k})|$ must be unbounded as $k \to \infty$.*

Lemma 5.1.2. *Consider (v_1, w_1) and (v_2, w_2) in P^Q such that*

$$h(q_{w_1} \circ q_{v_1}) \neq h(q_{w_2} \circ q_{v_2}).$$

Then any path in P^Q from (v_1, w_1) to (v_2, w_2) crosses infinitely many bones.

Proof. Consider an arbitrary path in P^Q from p_1 to p_2:

$$p : [0,1] \to P^Q \ , \ p(t) = (v(t), w(t))$$
$$p(0) = (v_1, w_1) \ , \ p(1) = (v_2, w_2).$$

For a fixed $k \in \mathbb{N}$, as t goes from 0 to 1, $\mathrm{Neg}((q_{w(t)} \circ q_{v(t)})^{\circ k})$ changes whenever a fixed point of negative type appears or disappears for $(q_{w(t)} \circ q_{v(t)}) \circ k$ (i.e., whenever a periodic point of negative type and period dividing k appears or vanishes for $q_{w(t)} \circ q_{v(t)}$). An existing negative-type fixed point of $(q_{w(t)} \circ q_{v(t)})^{\circ k}$ can be lost under continuous deformations of the map by becoming a positive-type fixed point. Conversely, such a fixed point can appear by a reverse process. Both changes imply the existence of an intermediate state, corresponding to some $t^* \in [0,1]$, in which the respective fixed point is a critical point of $(q_{w(t^*)} \circ q_{v(t^*)})^{\circ k}$. (In other words, a critical point of $q_{w(t^*)} \circ q_{v(t^*)}$ has to be periodic of period dividing k.) This implies that $p(t^*) = (v(t^*), w(t^*)) \in P^Q$ is on either a left or a right bone of period $2n$ dividing $2k$.

So if the integer $\mathrm{Neg}\left((q_{w(t)} \circ q_{v(t)})^{\circ k}\right)$ has an actual change at $t = t^*$, then the path $p(t)$ crosses a bone at $t = t^*$.

To end the proof of the lemma, suppose that the path $p(t)$ only crosses N bones. Then, for all $k \in \mathbb{N}$,

$$|\mathrm{Neg}((q_{w_2} \circ q_{v_2})^{\circ k}) - \mathrm{Neg}((q_{w_1} \circ q_{v_1})^{\circ k})|$$

would be bounded by N, a contradiction with Lemma 5.1.3. $\square$

5.2. **The entropy and the cellular structure.**

Recall that either parameter space, P^{ST} or P^Q, has for each fixed value of n an associated cellular complex structure, called P^{ST}_n and P^Q_n, respectively. The two cell complexes are homeomorphic through the function η defined in Section 2.11.

The following lemma is valid for either complexes, $P_n = P^{ST}_n$ or $P_n = P^Q_n$.

Lemma 5.2.1. *For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that, if p and p' are parameters that belong to the same closed cell in P_n and h_p represents the entropy value at the parameter p, then:*

$$|h_p - h_{p'}| < \epsilon.$$

Proof. Suppose the contrary: there exists $\epsilon > 0$ such that, for all $n \in \mathbb{N}$, there are two parameters p_n and p'_n in some common cell of P_n with

$$|h_{p_n} - h_{p'_n}| \geq \epsilon.$$

By the compactness of P, we can choose a subsequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that both $\{p_{k_n}\}_{n\in\mathbb{N}}$ and $\{p'_{k_n}\}_{n\in\mathbb{N}}$ converge in P:
$p_{k_n} \to p$ as $n \to \infty$ and $p'_{k_n} \to p'$ as $n \to \infty$.

The entropy function is a continuous function of parameters in either family (see for example [16]). We use this and take the limit:

$$|h_{p_{k_n}} - h_{p'_{k_n}}| \geq \epsilon \quad \Rightarrow \quad |h_p - h_{p'}| \geq \epsilon.$$

Moreover, the closed cells of P_n are nested as n increases (in other words, the cell complex gets "finer" with larger values of n).

Fix an arbitrary $N \in \mathbb{N}$. For all $k_n \geq N$, p_{k_n} and p'_{k_n} are in the same closed cell of P_{k_n}, and hence in the same closed cell of P_N.

In conclusion, for any arbitrary $N \in \mathbb{N}$, p and p' are in the same closed cell of P_N, yet

$$|h_p - h_{p'}| \geq \epsilon > 0,$$

a contradiction with Lemma 5.1.4. □

Lemma 5.2.2. *Fix $n \in \mathbb{N}$. In either parameter space, the entropy function restricted to any closed cell in P_n takes its maximum and minimum values on the boundary of the cell (more precisely on the boundary vertices).*

Proof. In the case $P_n = P_n^{ST}$, the proof is a simple corollary of Theorem 5.1.1. We have to prove the statement for $P_n = P_n^Q$.

For the fixed $n \in \mathbb{N}$, suppose the lemma is not true for some closed cell $C_n^Q \in P_n^Q$, that is, there exists $(v^*, w^*) \in int(C_n^Q)$ such that

$$h(q_{w^*} \circ q_{v^*}) > h_{max} \ ,$$

where h_{max} is the maximum value of the entropy on the boundary $\delta(C_n^Q)$.

Let

$$\epsilon = \frac{h(q_{w^*} \circ q_{v^*}) - h_{max}}{2} \geq 0.$$

By Lemma 5.2.1, there exists $m \in \mathbb{N}$ such that the entropy variation on all closed cells of P_m^Q is less than ϵ. WLOG, we can take $m > n$. Call C_m^Q the closed cell in P_m^Q such that $(v^*, w^*) \in C_m^Q \subset C_n^Q$ and consider any arbitrary vertex (v_m, w_m) of C_m^Q.

As $(v*, w^*)$ and (v_m, w_m) are in C_m^Q, we automatically have:

$$|h(q_{w^*} \circ q_{v^*}) - h(q_{w_m} \circ q_{v_m})| < \epsilon.$$

But $\quad h_{max} + 2\epsilon \leq h(q_{w^*} \circ q_{v^*})$, so:

$$h(q_{w_m} \circ q_{v_m}) > h_{max}.$$

The homeomorphism of complexes $\eta_m^{-1} : P_m^Q \longrightarrow P_m^{ST}$ carries vertices to vertices with the same entropy, edges to edges with the same interval of entropies, and 2-cells to 2-cells. So $C_m^{ST} = \eta_m^{-1}(C_m^Q)$ will be a 2-cell in P_m^{ST} and $(v_m^{ST}, w_m^{ST}) = \eta_m^{-1}(v_m, w_m)$ will be a vertex of C_m^{ST}. Also, $\eta_n^{-1}(\delta C_n^Q) = \delta(C_n^Q) = \delta C_n^{ST}$, so the maximum value

$h_{max}(\delta C_n^Q)$ of the entropy on δC_n^Q is the same as the maximum value $h_{max}(\delta C_n^{ST})$ on δC_n^{ST}. Hence we have

$$h(st_{w_m^{ST}} \circ st_{v_m^{ST}}) = h(q_{w_m} \circ q_{v_m}) > h_{max}(\delta C_n^Q) = h_{max}(\delta C_n^{ST}),$$

a contradiction, since the result has already been proved for P^{ST}. $\square$

Corollary 5.2.1. *For a fixed $n \in \mathbb{N}$, the interval of entropy values realized by any cell in P_n^Q is the same as the interval of values for the corresponding cell in P_N^{ST}.*

Definition 5.2.1. In either family, we call the h_0-*isentrope* the level-set of the entropy corresponding to the fixed value h_0.

$$i^{ST}(h_0) = \{(v,w) \in P^{ST} \ / \ h(st_w \circ st_v) = h_0\}$$

$$i^Q(h_0) = \{(v,w) \in P^Q \ / \ h(q_w \circ q_v) = h_0\}.$$

For a fixed $n \in \mathbb{N}^*$, we call $N_n^{ST}(h_0)$ the union of all cells C_n^{ST} in P_n^{ST} which intersect $i^{ST}(h_0)$, and we call $N_n^Q(h_0)$ the union of all cells C_n^Q in P_n^Q which intersect $i^Q(h_0)$.

Remarks : (1) Clearly: $i^{ST}(h_0) \subset N_n^{ST}(h_0)$ and $i^Q(h_0) \subset N_n^Q(h_0)$.

(2) Recall that for fixed n we have the homeomorphism of cell complexes:

$$\eta_n : P_n^{ST} \to P_n^Q.$$

If C_n^{ST} is a cell in P_n^{ST} that touches $i^{ST}(h_0)$, then the corresponding cell $C_n^Q = \eta_n(C_n^{ST})$ will touch $i^Q(h_0)$ and conversely. This follows from Corollary 5.2.3, which states that the interval of entropy values is the same in the two closed cells C_n^{ST} and C_n^Q.

Fix an entropy value $h_0 \in [0, \log 4]$ and an $n \in \mathbb{N}^*$. Since $N_n^{ST}(h_0)$ and $N_n^Q(h_0)$ are both unions of closed cells, they are compact subsets of P^{ST} and P^Q, respectively. By the previous theorem, $N_n^{ST}(h_0)$ is connected, so its image $N_n^Q(h_0) = \eta_n(N_n^{ST}(h_0))$ is also connected. Hence we have the following:

Summary. *For any $n \in \mathbb{N}^*$, the set $N_n^Q(h_0)$ is compact, connected and contains $i^Q(h_0)$.*

We have now a quite comprehensive description of the sets $N_n^Q(h_0)$. To obtain topological properties of $i^Q(h_0)$, we try to relate it to the collection $\{N_n^Q(h_0)\}_{n\in\mathbb{N}}$.

Lemma 5.2.3. *$i^Q(h_0)$ is the countable union of all $N_n^Q(h_0)$, with $n \in \mathbb{N}$.*

Proof. Since $i^Q(h_0) \subset N_n^Q(h_0)$ for all $n \in \mathbb{N}^*$, the inclusion $i^Q(h_0) \subset \bigcap N_n^Q(h_0)$ is trivial.

For the converse, suppose there exists $(v,w) \in \bigcap N_n^Q(h_0) \backslash i^Q(h_0)$. In other words: for any arbitrary $n \in \mathbb{N}^*$, (v,w) is contained in a closed cell $C_n^Q \subset P_n^Q$ that touches $i^Q(h_0)$, but such that $(v,w) \notin i^Q(h_0)$. For any such closed cell C_n^Q, there exists $(v_n^*, w_n^*) \in i^Q(h_0) \cap C_n^Q$.

The sequence $(v_n^*, w_n^*)_{n\in\mathbb{N}^*}$ satisfies in particular:

(1) $(v_n^*, w_n^*) \neq (v,w), \ \forall n \in \mathbb{N}^*$

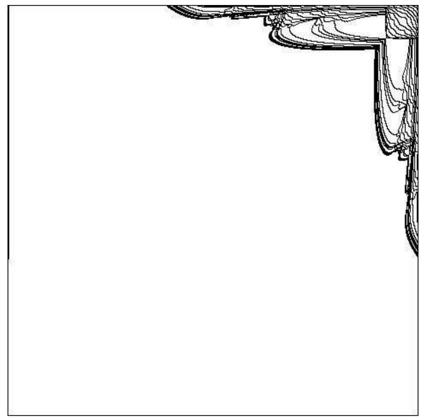


FIGURE 17. *The isentropes in P^Q appear to be either arcs joining two points in ∂P^Q, or connected regions between such arcs, or a single point (the case $(v,w)=(1,1)$ of entropy $\log 4$).*

(2) $h(q_{w_n^*} \circ q_{v_n^*}) = h_0$.

We calculate:

$$\mid h(q_{w^*} \circ q_{v^*}) - h(q_w \circ q_v) \mid = \mid h_0 - h(q_w \circ q_v) \mid .$$

This contradicts the statement of Lemma 5.2.1: the maximal variation of the entropy over cells in P_n^Q can be made arbitrarily small by increasing n. □

Theorem 5.2.1. *$i^Q(h_0)$, the h_0-isentrope in P^Q, is connected.*

Proof. $i^Q(h_0)$ is an intersection of compact, connected sets in P^Q; therefore it is compact and connected. □

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